A New Solution to the Random Assignment Problem with Private Endowment

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Abstract

This paper proposes a new mechanism to solve random assignment problems in which some agents have private endowments. The new mechanism generalizes the Probabilistic Serial mechanism by letting agents benefit from the popularity of their private endowments, which is summarized by the idea of "you request my house - I get your speed". Interestingly, the same idea can also be used to deal with indifferent preferences in Probabilistic Serial and our generalization. Our method is straightforward to understand and easy to implement compared with the previous method of solving maximum network flow problems in the literature.

Keywords: Random Assignment, Private Endowment, Probabilistic Serial, Ordinal Efficiency

JEL Classification: C71, C78, D71

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1 Introduction

In many matching markets the central question is how to assign indivisible objects to agents without using monetary transfers. Examples include the assignment of public school seats to children, the assignment of on-campus apartments to students, and the assignment of donated kidneys to patients. Depending on the ownership structure in the problems, they are often classified as *house allocation problem*, *house allocation problem with existing tenants* and *house exchange*. In a house allocation problem no object is exogenously owned by any agent, while in a house exchange problem every object is exogenously owned by some agent, and a house allocation problem with existing tenants is a mixture of the other two types. When some agents have private endowments, individual rationality is an important constraint.

In this paper we make two contributions to the literature. First, we propose a generalization of the well-known simultaneous eating algorithm Probabilistic Serial (PS) to solve house allocation problems with existing tenants when all agents have strict preferences. We denote our generalization by PS^E. Second, when agents can have indifferent preferences, we propose a new method to adapt PS as well as PS^E to indifferent preferences. Our method is more straightforward to understand and easier to implement than the current method in the literature. It is interesting that both of our contributions are driven the same idea we call "you request my house - I get your speed". In the following we elaborate on each contribution.

Bogomolnaia and Moulin (2001) propose PS to solve house allocation problems. PS is implemented in a very simple way: imagine each object as a "divisible cake" and let agents "eat" objects according to their preference orderings with the same speed of one; each agent's consumption is the random assignment he obtains in PS. Compared to previous mechanisms such as Random Priority (RP, Abdulkadiroğlu and Sönmez, 1998), the most important property of PS is *ordinal efficiency*. It is a desirable efficiency notion between ex ante efficiency[1] and ex post efficiency (RP is only ex post efficient). However, if an agent has a private endowment, he may obtain a positive fraction of an object worse than his private endowment in PS. So PS is not *individually rational* for house allocation problems with existing tenants.

[1]When the cardinal utilities of agents are known, Hylland and Zeckhauser (1979) use the peseudo competitive market to obtain ex ante efficient random assignments.

Our proposed PS^E deviates from PS in two aspects. First, at any time of the procedure if the private endowment of agent is eaten by some other agents, the agent can instantly get an additional eating speed, which is equal to the total speed at which his private endowment is being eaten. This is what we mean "you request my house - I get your speed". Second, at any time of the procedure if several existing tenants (those with private endowments) want to consume each other's private endowment such that they form a cycle, we let them trade the fractions of their private endowments instantly. This is similar to the Top Trading Cycle (TTC) mechanism proposed by Shapley and Scarf (1974). But in the paper we show that this aspect is also the outcome of "you request my house - I get your speed". Both aspects guarantee that the demand of an existing tenant must be satisfied weakly before his private endowment is exhausted. So PS^E must be *individually rational*. PS^E is *ordinally efficient* since it is a simultaneous eating algorithm.[2] PS^E is also *envy-free among new agents* (those without private endowments) since they always have the same eating speed of one.

The motivation behind PS^E is not just to satisfy individual rationality; indeed there are infinitely many generalizations of PS to satisfy individual rationality. We argue that PS^E is a natural generalization of PS by fully allowing existing tenants to trade their private endowments with the others. To illustrate it, if we view the objects without owners (we call *social endowments*) as owned by all agents collectively, and require that in an eating algorithm each agent's eating speed come from the transfer of his endowments (either social or private) as implied by "you request my house - I get your speed", then we prove that the eating algorithm must be PS or PS^E depending on whether there are existing tenants. Hence both PS and PS^E are dynamic processes of trading eating speeds (ownerships).

In another way to support the above argument we prove that PS^E generalizes two equivalence theorems of PS in the literature, which are summarized as follows.

$$\begin{tikzcd}
PS \arrow[r, Leftrightarrow, "\text{Kesten (2009)}"] \arrow[d, "\text{this paper}"] & TTCfED \arrow[d, "\text{this paper}"] \\
PS^E \arrow[r, Leftrightarrow, "\text{this paper}"] & TTC^E
\end{tikzcd}
\qquad
\begin{tikzcd}
PS \arrow[r, Leftrightarrow, "\text{Che and Kojima (2010)}", "\text{asym.}"'] \arrow[d, "\text{this paper}"] & RP \arrow[d, "\text{Abdulkadiroğlu and Sönmez (1999)}"] \\
PS^E \arrow[r, Leftrightarrow, "\text{this paper}", "\text{asym.}"'] & \text{random YRMH-IGYT}
\end{tikzcd}$$

[2]When existing-tenant cycles are traded instantly, it is as if that existing tenants in the cycles have infinitely large eating speeds.

Here "$a \Leftrightarrow b$" means that a is equivalent to b, "$a \underset{\text{asym.}}{\Leftrightarrow} b$" means that a is asymptotically equivalent to b, and "$a \to b$" means that a is generalized to b. Specifically, Kesten (2009) proposes $TTCfED$ for house allocation problems. It proceeds by first assigning the fractions of all houses equally to all agents, then letting agents trade their fractional endowments as in TTC. Kesten proves that PS is equivalent to $TTCfED$. We show that this result still holds between PS^E and a direct generalization of $TTCfED$ we call TTC^E.[3] Che and Kojima (2010) prove that PS is asymptotically equivalent to RP. When there are private endowments, Abdulkadiroğlu and Sönmez (1999) generalize RP to the *random "you request my house-I get your turn" (YRMH-IGYT)* mechanism. By using the technique of Che and Kojima we similarly prove that PS^E is asymptotically equivalent to random YRMH-IGYT. The intuition behind it is that the additional eating speeds of existing tenants in PS^E correspond to their additional chances to request objects in random YRMH-IGYT in large markets.

As a mechanism we hope PS^E has a good incentive property. However, Bogomolnaia and Moulin (2001) has proved the incompatibility between ordinal efficiency, strategy-proofness and a weaker fairness notion called equal treatment of equals.[4] Nesterov (2015) prove more impossibility theorems for random assignment problems. Hence PS^E must not be strategy-proof. But we prove that it satisfies a property called *bounded invariance*. Bogomolnaia and Heo (2012) use this property, along with ordinal efficiency and envy-freeness, to characterize PS. This property implies that PS^E cannot be weakly manipulated by a class of strategies called *dropping strategies*, and cannot be manipulated by *truncation strategies*. Truncation strategy is an important manipulation strategy that has been well studied in the literature (e.g., Roth and Rothblum, 1999; Ehlers, 2008; Coles and Shorrer, 2014; Castillo and Dianat, 2016).

We also compare PS^E with the other mechanisms in the literature. In particular, Yilmaz (2010) proposes another generalization of PS called *Individually Rational Probabilistic Serial* (denoted by PS^{IR}). PS^{IR} minimally deviates from PS to accommodate the individual rationality constraint. Hence PS^{IR} satisfies a stronger fairness notion (i.e., no justified-envy) than PS^E. However, PS^{IR} has a worse incentive property than PS^E since it is unknown to be immune to any class of strategies. In particular, it can be

[3] In TTC^E only the fractions of social endowments are equally assigned to all agents.

[4] Equal treatment of equals says that if any two agents report the same preferences, they obtain the same random assignment in the mechanism.

manipulated by truncation strategies. We have more discussion about their difference in Section 6. In a not circulated paper Sethuraman (2001) proposes a generalization of PS called *uniform eating rate* (UER). According to the description of Yılmaz (2010), UER is close to PS^E. But UER lets an existing tenant's eating speed be one as long as his individual rationality is not to be violated; otherwise, UER lets his eating speed equal the total speed at which his private endowment is being eaten. Other than to satisfy individual rationality there is no specific reason for the speed setting in UER or any characterization of it. But there are infinitely may ways to satisfy individual rationality. To show the uniqueness of PS^E we provide a characterization and two equivalence theorems. Random YRMH-IGYT is a desirable mechanism because it is strategy-proof. But it has weaker efficiency and fairness properties than PS^E. So the choice between them depends on the criterion used in a specific application. Also remember that they can be close to each other in large markets.

In our second contribution, we show that the idea of "you request my house - I get your speed" can easily solve the difficulty caused by indifferent preferences in PS and PS^E. In the literature Katta and Sethuraman (2006) transform an allocation problem into a network problem, and show that indifferent preferences in PS can be solved by iteratively solving a maximum network flow problem. This method is further used by Yılmaz (2009) to deal with indifferent preferences in PS^{IR}, and used by Athanassoglou and Sethuraman (2011) to deal with indifferent preferences in their generalization of PS^{IR} to allocation problems with fractional endowments. By contrast, our method is straightforward to understand and easy to implement.

To see the difficulty caused by indifference, suppose two agents i, j have preferences over two objects h, h' such that $h \sim_i h'$ and $h \succ_j h'$. If after arbitrarily breaking preference ties i uses the preference relation $\succ'_i$ such that $h \succ'_i h'$, then in PS both i, j will first eat h then h'. But the random assignment found in this way is not ordinally efficient since i, j can exchange an equal fraction of their consumed h and h' such that j is strictly better off and i is not worse off. Our method to solve this problem is as follows. When h is exhausted in PS, since h' remains and $h \sim_i h'$, we let i label his consumption of h as "available" for the other agents to consume. Since j strictly prefers h to h', j will eat i's available consumption of h. Then we compensate i by letting i eat h' with an additional speed that equals the speed at which the other agents eat i's consumption of h. In a house allocation problem with existing tenants, if h' happens to be j's private endowment, then i and j form a cycle when j wants to consume i's consumption of h. Then we trade the cycle immediately. So in either way i's welfare is always exactly compensated. In the paper we formalize this idea to propose an adaptation of PS^E to indifferent preferences. When there are no private endowments, we essentially also propose an adaption of PS to indifferent preferences.

Related Literature There are a lot of papers studying random assignment problems and PS. We discuss some of them here. Specifically, since PS is not strategy-proof, Ekici and Kesten (2015) use multiple equilibrium solutions to study the possible outcome of PS. They show that PS may not have the nice properties proved by Bogomolnaia and Moulin (2001) in an (ordinal) Nash equilibrium. Hugh-Jones et al. (2014) use laboratory experiments to study the incentive property of PS. They also find that misreporting is a significant problem in PS. Dogan et al. (2016) propose a new efficiency criterion for random assignments with only the information of ordinal preferences. By the criterion the outcome of PS can be improved in efficiency without sacrificing fairness. Hashimoto et al. (2014) and Bogomolnaia (2015) respectively provide two other characterizations of PS.

Several papers have extended PS to the other environments. Kojima (2009) extends PS to the environment in which agents of equal demands demand more than one objects. Heo (2014) does a similar extension, but she allows agents to have heterogeneous demands. Budish et al. (2013) extend PS to the environment in which agents have multi-item demands and random assignments are subject to exogenous constraints. They show that only a special structure of contraints can be dealt with. Balbuzanov (2014) extends PS to kidney exchange problems in which the length of trading cycles is constrained.

Kesten and Ünver (2015) is different from the above papers in that they extend the Deferred Acceptance algorithm (DA, Gale and Shapley, 1962) to solve a random assignment problem in which objects have coarse priority rankings over agents. A house allocation problem with existing tenants can be seen as a special case of the problem. Their adaption of DA is different from PS in a house allocation problem.

The rest of the paper is organized as follows. Section 2 describes the assignment problem and defines some concepts. Section 3 proposes PS^E under the strict preferences environment. Section 4 discusses the properties of PS^E. Section 5 proves the two equivalence theorems. Section 6 compares PS^E with the other mechanisms. Section 7 presents our method of dealing with indifferent preferences. Section 8 concludes. All proofs are in Appendix.

2 House Allocation Problem with Existing Tenants

2.1 The Model

A *house allocation problem with existing tenants* is a four-tuple $m = \{I, H, \pi, \succsim_I\}$ where I is a finite set of agents, H is a finite set of houses, $\pi : I \to H$ is an endowment function, and $\succsim_I = (\succsim_i)_{i \in I}$ is the preference profile of all agents. There is a null house h_0 in H such that if $\pi(i) = h_0$, then it means that i has no private endowment. Each non-null house has only one copy and can be owned by at most one agent. So $\pi(i) \neq \pi(j)$ for all distinct $i, j \in I$ unless $\pi(i) = \pi(j) = h_0$. The agents who own non-null houses are called *existing tenants*, and their private endowments are called *occupied houses*. The set of existing tenants is denoted by I_E and the set of occupied houses is denoted by H_O. The remaining agents and houses are respectively called *new agents* and *vacant houses*, and their sets are denoted by I_N and H_V respectively. For convenience we also call H_V *social endowments*. Each agent i demands one house and has a preference relation $\succsim_i$ over all houses H with the strict part denoted by $\succ_i$. We do not require $\succsim_i$ be strict, but we require that no agent be indifferent between a real house $h \in H \backslash \{h_0\}$ and the null house h_0.[5] Every house h satisfying $h \succsim_i \pi(i)$ is *acceptable* to i. When H, I, and π are clear in the context, we sometimes denote a problem simply by $\{\succsim_I\}$. Let $\mathcal{M}$ be the set of all house allocation problems with existing tenants, and $\mathcal{R}$ be the set of all preference relations over H.

2.2 Random Assignment and Other Concepts

A *random assignment* is a matrix $q = (q_{ih})_{i \in I, h \in H}$ such that $\sum_{i \in I} q_{ih} \leq 1$ for $\forall h \in H \backslash \{h_0\}$ and $\sum_{h \in H} q_{ih} = 1$ for $\forall i \in I$. Here q_{ih} is the probability that i is assigned the house h and q_{ih_0} is the probability that i is not assigned any house. So $q_i = (q_{ih})_{h \in H}$ is the lottery i obtains. If $q_{ih} \in \{0, 1\}$ for all $i \in I$ and all $h \in H$, then q is a *deterministic assignment*.

[5]That is, h_0 denotes the status of being unassigned while every agent strict prefers to have a house; otherwise he does not need to participate in the allocation problem.

A lottery q_i is *individually rational* for agent i if $h \succsim_i \pi(i)$ for all $h \in H$ such that $q_{ih} > 0$. That is, i is never assigned an unacceptable house with a positive probability. Then a random assignment q is *individually rational* if q_i is individually rational for every agent i. Given $\succsim_i$, we can compare any two lotteries assigned to i in the sense of first-order stochastic dominance. Formally, a lottery q_i *first-order stochastically dominates* another lottery q'_i, denoted by $q_i \trianglerighteq_i q'_i$, if

$$\sum_{h' \succsim_i h} q_{ih'} \geq \sum_{h' \succsim_i h} q'_{ih'} \text{ for } \forall h \in H.$$

If the above inequality holds strictly for an acceptable h, we say q_i strictly stochastically dominates q'_i and denote it by $q_i \triangleright_i q'_i$. For any two random assignments q and q', we say q strictly stochastically dominates q' if $q_i \trianglerighteq_i q'_i$ for all i and $q_j \triangleright_j q'_j$ for some j. We denote it by $q \triangleright q'$. Then a random assignment q is *ordinally efficient* if there does not exist q' such that $q' \triangleright q$.

In a random assignment q we say an agent i *envies* another agent j if $q_i \trianglerighteq_i q_j$ does not hold, and i *weakly envies* j if $q_j \triangleright_i q_i$ holds. Then we say q is *envy-free* if any i does not envy any other j, and is *weakly envy-free* if any i does not weakly envy any other j. Similarly, we say q is *new-agent envy-free* if any new agent i does not envy any other new agent j, and q is *weakly new-agent envy-free* if any new agent i does not weakly envy any other new agent j.

For every problem $m \in \mathcal{M}$, let $Q(m)$ be the set of all random assignments for m. Let $\mathcal{Q} := \bigcup_{m \in \mathcal{M}} Q(m)$ be the set of all possible random assignments. Then a *random assignment mechanism* is a function $\varphi : \mathcal{M} \to \mathcal{Q}$ such that $\varphi(m) \in Q(m)$ for $\forall m \in \mathcal{M}$. In the above paragraphs we have defined multiple properties of a random assignment. We say a mechanism φ has one of these properties if for all $m \in \mathcal{M}$, $\varphi(m)$ has the property. In addition, we say φ is *boundedly invariant* if any agent i in any problem reports a preference relation $\succsim'_i$ which coincides with his true preference relation $\succsim_i$ at all houses weakly better than h, then the assignments of all houses weakly better than h in $\succsim_i$ do not change. Formally, let $U(\succsim_i, h) := \{h' \in H \mid h' \succsim_i h\}$ be the upper contour set of $\succsim_i$ at h and $\succsim_i |_{U(\succsim_i, h)}$ be the restriction of $\succsim_i$ to $U(\succsim_i, h)$, then for $\forall \succsim'_i \in \mathcal{R}$ such that $U(\succsim'_i, h) = U(\succsim_i, h)$ and $\succsim'_i |_{U(\succsim'_i, h)} = \succsim_i |_{U(\succsim_i, h)}$, we have $\varphi_{jh'}(\{\succsim_I\}) = \varphi_{jh'}(\{\succsim'_i, \succsim_{-i}\})$ for $\forall j \in I$ and $\forall h' \in U(\succsim_i, h)$. Finally, we say φ is *strategy-proof* if $\varphi_i(\{\succsim_I\}) \trianglerighteq_i \varphi_i(\{\succsim'_i, \succsim_{-i}\})$ for $\forall \succsim'_i \in \mathcal{R}$, $\forall i \in I$ and $\forall m = \{\succsim_I\}$. We say φ is *weakly strategy-proof* if for

$\forall i \in I$ there does not exist $\succsim_i' \in \mathcal{R}$ such that $\varphi_i(\succsim_i', \succsim_{-i}) \rhd_i \varphi_i(\{\succsim_I\})$.

3 The PS^E Mechanism

From this section to Section 6 we assume that all agents' preferences are strict. We deal with weak preferences in Section 7. As mentioned before, PS^E is generalization of PS with two new features. First, at any time $t \in [0, 1]$, if an existing tenant's private endowment is being eaten by the other agents, the existing tenant can immediately get an additional eating speed which equals the total speed at which his private endowment is being eaten. We call this feature "you request my house - I get your speed". Second, at any time $t \in [0, 1]$, if several existing tenants want to consume each other's private endowment such that they form a cycle, they trade the fractions of their private endowments instantly. How much can be traded depends on the remainder of each private endowment and the residual demand of each existing tenant in the cycle. The second feature is similar to TTC, but PS^E can still be seen as a simultaneous eating algorithm[6] defined by Bogomolnaia and Moulin (BM hereafter): when there are cycles among some existing tenants, let their eating speeds be infinitely large. Before giving the formal definition we first illustrate PS^E through a simple example.

3.1 An Example

Example 1. *A problem consists of $H = \{h_0, h_1, \ldots, h_6\}$ and $I = \{i_1, i_2, \ldots, i_6\}$. Here i_1, i_2, i_3, i_4, i_5 are existing tenants and own h_1, h_2, h_3, h_4, h_5 respectively. i_6 is a new agent and h_6 is a vacant house. The following table is the preference profile of all agents where $\succsim_o$ is i_o's preference list. Boxed houses are private endowments of the corresponding agents. Unacceptable houses for existing tenants are omitted from their preference lists.*

PS^E will solve this problem in the following steps.

Step 1: *At $t = 0$, i_1, i_4 want to consume h_2, i_2, i_6 want to consume h_3, and i_3, i_5 want to consume h_1.*

There is a cycle consisting of existing tenants i_1, i_2, i_3 and their private endowments. Trade this cycle instantly. Since every private endowment in the cycle is never

[6] Formally, a *simultaneous eating algorithm* is identified by a profile of eating speed functions $\{s_i(t)\}_{i \in I}$ where $s_i(t) : [0, 1] \to \mathbb{R}_+$ is a measurable function such that $\int_0^1 s_i(t)dt = 1$.

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$
h_2	h_3	h_1	h_2	h_1	h_3
h_3	$\boxed{h_2}$	h_5	h_6	h_6	h_4
$\boxed{h_1}$		$\boxed{h_3}$	h_5	h_4	h_5
			$\boxed{h_4}$	$\boxed{h_5}$	h_0

consumed and every existing tenant in the cycle demands one house, we trade one unit of house in the cycle. After the trade i_1, i_2, i_3 own h_2, h_3, h_1 respectively and they stop consuming other houses. Note that after the trading the time is still at $t = 0$ since the trading happens instantly.

Step 2: *Still at $t = 0$, i_4, i_5 want to consume h_6 and i_6 wants to consume h_4.*

There is no cycle. i_6 eats h_4 with speed one. i_5 eats h_6 also with speed one. But i_4 eats h_6 with speed two since his private endowment h_4 is being eaten with a total speed of one.

At $t = 1/3$, h_6 is exhausted. Then i_4 consumes $2/3$ of h_6, i_5 consumes $1/3$ of h_6, and i_6 consumes $1/3$ of h_4

Step 3: *At $t = 1/3$, i_4 wants to consume h_5 and i_5, i_6 want to consume h_4.*

There is a cycle consisting of existing tenants i_4, i_5 and their private endowments. h_4 has a remainder of $2/3$, h_5 has remainder of 1, while i_4's remaining demand is $1/3$ and i_5's remaining demand is $2/3$. So we trade $1/3$ of each house in the cycle. After the trade i_4 gets $1/3$ of h_5 and stops consuming other houses, and i_5 gets $1/3$ of h_4. Note that the time is at $t = 1/3$.

Step 4: *Still at $t = 1/3$, both i_5 and i_6 want to consume h_4. There is no cycle, so each of them eats h_4 with speed one. At $t = 1/2$, h_4 is exhausted. Then each of i_5 and i_6 gets $1/6$ of h_4.*

Step 5: *At $t = 1/2$, both i_5 and i_6 want to consume h_5. Since h_5 is the private endowment of i_5, we say there is a i_5's self-cycle. Since h_5 has a remainder of $2/3$ and i_5's remaining demand is $1/2$, we trade $1/6$ of h_5 in the cycle. After the trade i_5 gets $1/6$ of h_5 and stops consuming other houses.*

Step 6: *Still at $t = 1/2$, i_6 is the only agent, so he eats h_5 with speed one. At $t = 1$, h_5 is exhausted and i_6's remaining demand is filled.*

The above steps are summarized in Table 1.

Step d: what happened	i_1	i_2	i_3	i_4	i_5	i_6
1: $i_1 - i_2 - i_3 - i_1$ cycle	h_2	h_3	h_1			
2: eating				$2/3h_6$	$1/3h_6$	$1/3h_4$
3: $i_4 - i_5 - i_4$ cycle				$1/3h_5$	$1/3h_4$	
4: eating					$1/6h_4$	$1/6h_4$
5: i_5's self-cycle					$1/6h_5$	
6: eating						$1/2h_5$

Table 1: The procedure of PS^E in Example 1.

3.2 Formal Definition

Now we are ready to present the formal definition of PS^E. As in the above example we track the procedure of PS^E by discrete steps at which some houses are exhausted or some agents' demands are filled. The following are useful some notations.

d: steps; t_d: the time at which step d ends;
$r_i(d)$: i's residual demand when step d ends; $r_h(d)$: remainder of h when step d ends;
$s_i(t)$: i's eating speed at t; $s_h(t)$: total speed at which h is being eaten at t;
$A_h(t)$: set of agents who point to h at t;
$H(d)/I(d)/I_N(d)/I_E(d)$: remaining houses/agents/new agents/existing tenants when step d ends.

Initialization: $I(0) = I$, $H(0) = H$, $r_i(0) = 1$ for $\forall i \in I$, $r_h(0) = 1$ for $\forall h \in H\backslash\{h_0\}$, $t_0 = 0$.

Step $d \geq 1$: If $I(d-1) = \emptyset$ or $H(d-1) = \emptyset$, stop. Otherwise, proceed to the Pointing stage.

- *Pointing*: Every $i \in I(d-1)$ points to his most preferred house in $H(d-1)$. If it is h_0, let i point to his own copy of h_0. Every occupied house in $H(d-1)$ point to its owner if its owner is in $I(d-1)$. Go to the Consuming stage.

- *Consuming*: If there exist cycles consisting of existing tenants and their private endowments such as $h_1 \to i_1 \to h_2 \to i_2 \to \cdots \to i_k \to h_1$, go to the Trading Cycle stage. Otherwise, go to the Eating stage.

  - *Trading Cycle*: Trade the above cycles instantly. For every cycle c, the trading quota of c is defined as

    $$TQ(c) := \min\{\min_{i \in I(c)} r_i(d-1), \min_{h \in H(c)} r_h(d-1)\},$$

    where $I(c)$ and $H(c)$ are the set of existing tenants and the set of occupied houses involved in c respectively. So every agent in $I(c)$ obtains $TQ(c)$ of the house he points to. Then $r_h(d) := r_h(d-1) - TQ(c)$ for $\forall h \in H(c)$ and $r_i(d) := r_i(d-1) - TQ(c)$ for $\forall i \in I(c)$.

    For every house h and every agent i not involved in any cycle, $r_h(d) := r_h(d-1)$ and $r_i(d) := r_i(d-1)$. Agents in $G(d) := \{i \in I(d-1) : r_i(d) = 0\}$ are full. Then $I(d) := I(d-1) \backslash G(d)$ and $H(d) := H(d-1) \backslash \{h \in H(d-1) : r_h(d) = 0\}$. Step d ends at time $t_d := t_{d-1}$. Go to step $d+1$.

  - Eating: All agents simultaneously eat the houses they point to with eating speeds specified as follows.

    For $t \geq t_{d-1}$, $s_i(t) := 1$ for $\forall i \in I_N(d-1)$, and $s_j(t) := s_{\pi(j)}(t) + 1$ for $\forall j \in I_E(d-1)$ where $s_{\pi(j)}(t) := \sum_{i \in A_{\pi(j)}(t_{d-1})} s_i(t)$ is the total speed at which $\pi(j)$ is being eaten.

    Define $t_d := \min\{t_a : r_a(d-1) - s_a(t)(t_a - t_{d-1}) = 0, a \in H(d-1) \cup I(d-1)\}$. That is, step d ends when a house in $H(d-1)$ is exhausted or an agent in $I(d-1)$ is full, depending on which happens earlier.

    Then $r_h(d) := r_h(d-1) - s_h(t_{d-1})(t_d - t_{d-1})$ for $\forall h \in H(d-1)$ and $r_i(d) := r_i(d-1) - s_i(t_{d-1})(t_d - t_{d-1})$ for $\forall i \in I(d-1)$. Agents in $G(d) := \{i \in I(d-1) : r_i(d) = 0\}$ are full. Then $I(d) := I(d-1) \backslash G(d)$ and $H(d) := H(d-1) \backslash \{h \in H(d-1) : r_h(d) = 0\}$. Go to step $d+1$.

3.3 Characterization of PS^E

Although we call the first feature of PS^E "you request my house - I get your speed", its second feature is also the outcome of the idea. In this subsection we actually show that

PS^E is characterized by the idea among all simultaneous eating algorithms.[7] Specifically, by treating social endowments as equally owned by all agents collectively, we say a simultaneous eating algorithm satisfies "you request my house - I get your speed" if each agent's eating speed at any time comes from the transfers of their endowments (either social or private). Here we require that each social endowment uniformly transfer the total speed at which it is being eaten to its every owner, and each private endowment transfer the total speed at which it is being eaten exclusively to its unique owner. But if the owner of a private endowment is satisfied at any time, the private endowment is treated as a social endowment owned by the remaining agents.

Formally, at any $t \in [0,1]$ let $E_i(t)$ be the set of endowments of each agent i, which includes social endowments and his private endowment. Let $O_h(t)$ be the set of owners of each house h. $A_h(t)$, $s_i(t)$ and $s_h(t)$ have the same meanings defined before. Then a simultaneous eating algorithm satisfies "you request my house - I get your speed" if for each remaining h and each remaining i at t,

$$(1) \quad s_h(t) = \textstyle\sum_{i \in A_h(t)} s_i(t); \qquad (2) \quad s_i(t) = \textstyle\sum_{h \in E_i(t)} s_h(t)/|O_h(t)|.$$

It is interesting to observe that (1) and (2) are similar to market equilibrium conditions: if we imagine each agent's eating speed as his budget, then at any time t each agent spends his budget on his most preferred house; on the other hand, each agent's budget comes from the money the other agents spend on his endowments. Hence (1) and (2) characterize an equilibrium of trading eating speeds.

Proposition 1. *A simultaneous eating algorithm satisfies "you request my house - I get your speed" if and only if it is equivalent to PS^E.*

Here by saying two mechanisms are equivalent we mean they always find the same assignment for the same problem. The intuition behind the proof is rather simple. At any time t, if there is no cycle among existing tenants we can normalize the eating speed of every new agent i, which is equal to $\sum_{h \in E_i(t)} s_h(t)/|O_h(t)|$, to one. Then the eating speed of every existing tenant j is $s_j(t) = s_{\pi(j)}(t) + \sum_{h \in E_i(t)} s_h(t)/|O_h(t)| = s_{\pi(j)}(t) + 1$. If there are cycles, let a typical cycle be $\pi(j_1) \to j_1 \to \cdots \to \pi(j_n) \to j_n \to \pi(j_1)$. Then conditions (1) and (2) implies that $s_{j_1}(t) \leq \cdots \leq s_{j_n}(t) \leq s_{j_1}(t)$. So $s_{j_1}(t) \cdots = s_{j_n}(t)$.

[7] Bogomolnaia and Heo (2012) have shown that every random assignment can be seen as the outcome of a simultaneous eating process. So our characterization is essentially among all mechanisms.

However we know that $s_{j_2}(t) = s_{\pi(j_2)}(t) + \sum_{h \in E_i(t)}(s_h(t)/|O_h(t)|) \geq s_{j_1}(t) + s_i(t)$. So $s_i(t) = 0$ for every new agent i. Hence for any existing tenant j who is not involved in any cycle, $s_{\pi(j)} = 0$, which implies $s_j(t) = s_{\pi(j)}(t) + s_i(t) = 0$. So it is equivalent to trading the cycle instantly.

It is easy to see that when there are no private endowments, conditions (1) and (2) also characterize PS.

4 The Properties of PS^E

In this section we discuss the properties of PS^E in terms of efficiency, fairness and manipulability. First, since PS^E is a simultaneous eating algorithm, it must be *ordinally efficient*. Second, PS^E is obviously individually rational since existing tenants are satisfied no later than their private endowments are exhausted. Third, although PS^E does not satisfy envy-freeness, it satisfies *new-agent envy-freeness* since new agents always have the same eating speed.

Proposition 2. *The PS^E mechanism is ordinally efficient, individually rational and new-agent envy-free.*

BM prove that PS is weakly strategy-proof. However, the following example shows that PS^E is not.

Example 2. *A problem consists of $H = \{h_0, h_1, \ldots, h_9\}$ and $I = \{i_1, i_2, \ldots, i_8\}$. Here i_1, i_2 own h_1, h_2 respectively. The preference profile is as follows. $\succsim_o$ is the true preference relation of agent i_o. Agents i_3, i_4 have identical preferences, while i_5, i_6, i_7 have identical preferences.*

$\succsim_1$	$\succsim_2$	$\succsim_3$ / $\succsim_4$	$\succsim_5$ / $\succsim_6$ / $\succsim_7$	$\succsim_8$	$\succsim'_8$
h_7	h_7	h_1	h_2	h_1	h_1
h_8	h_9	h_3	h_3	h_8	h_3
$\boxed{h_1}$	$\boxed{h_2}$	h_4	h_4	h_3	h_4
		h_5	h_5	h_4	h_8
		h_6	h_6	h_5	h_5
		h_0	h_0	h_6	h_6
				h_0	h_0

When all agents report their true preferences, the procedure of PS^E and the assignment it finds are shown by Table 2.

time	i_1	i_2	i_3/i_4	$i_5/i_6/i_7$	i_8
$1/8$	$1/2h_7$	$1/2h_7$	$1/8h_1$	$1/8h_2$	$1/8h_1$
$+1/8$	$1/2h_8$	$1/2h_9$	$1/8h_1$	$1/8h_2$	$1/8h_1$
$+1/7$			$1/7h_3$	$1/7h_3$	$1/7h_8$
$+1/7$			$1/7h_4$	$1/7h_4$	$1/7h_8$
$+1/7$			$1/7h_5$	$1/7h_5$	$1/7h_8$
$+1/14$			$1/14h_6$	$1/14h_6$	$1/14h_8$
$+1/16$			$1/16h_6$	$1/16h_6$	$1/16h_6$

Table 2: The procedure of PS^E in Example 2

At $t = 1/4$, h_1 is exhausted and i_1, i_2 are full and stop consuming. Among the remaining agents only i_8 wants to consume h_8. But the remaining $1/2h_8$ can not satisfy the residual demand of i_8 which is $3/4$. At this point if i_8 strategically chooses to eat h_3, then eats h_4 at $t = 3/8$ and returns to h_8 at $t = 1/2$, his residual demand can be exactly satisfied by $1/2h_8$. So by using this strategy i_8 can eat more fractions of h_3 and h_4 without losing any fraction of h_8. So i_8 can manipulate PS^E by reporting $\succsim_8'$, and PS^E is not weakly strategy-proof.

In the above example i_8 manipulates PS^E by reshuffling his preferences. This kind of manipulation is inevitable in PS^E since some existing tenants may leave the algorithm earlier than the others. However, we prove that PS^E is *boundedly invariant*. This property means that the temporary assignment at any step of PS^E is determined only by the preferences that agents have revealed by the step. This implies that PS^E cannot be manipulated by some class of manipulation strategies. Specifically, we say $\succsim_i'$ is a *dropping strategy* of $\succsim_i$ if it is obtained by dropping some houses from the set of acceptable houses in $\succsim_i$. Formally, we require that $U(\succsim_i', \pi(i)) \subsetneqq U(\succsim_i, \pi(i))$ and $\succsim_i' |_{U(\succsim_i', \pi(i))} = \succsim_i |_{U(\succsim_i', \pi(i))}$. Then we say a mechanism φ is *weakly dropping-strategy-proof* if $\varphi(\{\succsim_{-i}, \succsim_i'\}) \rhd_i \varphi(\{\succsim_I\})$ does not hold for any i and any dropping strategy $\succsim_i'$ in any problem $\{\succsim_I\}$. We prove that PS^E is weakly dropping-strategy-proof.

A special case of dropping strategy is truncation strategy. Simply speaking, $\succsim_i'$ is a truncation strategy of $\succsim_i$ if it is obtained by moving $\pi(i)$ up in $\succsim_i$. Formally, there exists some $h \succsim_i \pi(i)$ such that $\succsim'_i |_{U(\succsim'_i,\pi(i))} =\succsim_i |_{U(\succsim_i,h)}$. We say a mechanism φ is *truncation-strategy-proof* if $\varphi_i(\{\succsim_I\}) \trianglerighteq_i \varphi_i(\{\succsim'_i, \succsim_{-i}\})$ for any i and any truncation strategy $\succsim'_i$ in any problem $\{\succsim_I\}$. We prove that PS^E is truncation-strategy-proof.

Proposition 3. *The PS^E mechanism is boundedly invariant, weakly dropping-strategy-proof and truncation-strategy-proof.*

5 Equivalence Theorems

In this section we prove two equivalence theorems of PS^E, both generalizing the counterparts of PS. In the first theorem we prove that PS^E is equivalent to a probabilistic version of TTC, which generalizes a result of Kesten (2009). In the second theorem we prove that PS^E is asymptotically equivalent to random YRMH-IGYT, which generalizes the result of Che and Kojima (2010). We discuss the two theorems one by one.

When there are no private endowments, Kesten proves that PS is equivalent to TTC *from Equal Division* ($TTCfED$). $TTCfED$ proceeds in two steps: at the first step the fractions of all vacant house are uniformly assigned to all agents such that each agent has the same endowment profile; at the second step agents trade their fractional endowments as in TTC under some regulation. In this paper we repeat the two steps except that existing tenants have private endowments. We call the corresponding mechanism TTC *from Equal Division of Social Endowments* and denote it by TTC^E. Its formal definition is presented below.

TTC from Equal Division of Social Endowments

$d/I(d)/H(d)/r_i(d)$: have the same interpretations as in PS^E;

$e_i(d)$: the endowment profile of agent i when step d ends;

$\{i_h \cdot \Delta h : \forall \Delta h \in e_i(d)\}$: set of pseudo-agents representing i when step d ends. Here i_h is the pseudo-agent holding Δh.[8] i is called i_h's host. Pseudo-agents that hold vacant houses are also called new pseudo-agents, and pseudo-agents that hold occupied houses are also called pseudo-tenants.

[8]For example, if $e_i(d) = \{h_1, 1/2h_2, 1/3h_3\}$, then i is represented by three pseudo-agents: $i_{h_1} \cdot h_1$, $i_{h_2} \cdot 1/2h_2$ and $i_{h_3} \cdot 1/3h_3$.

Initialization: $I(0) = I$, $H(0) = H$, $e_i(0) = \{\frac{1}{|I|}h\}_{h \in H_V} \cup \{\pi(i)\}$, and $r_i(0) = 1$ for $\forall i \in I(0)$.

Step $d \geq 1$: If $I(d-1) = \emptyset$ or $H(d-1) = \emptyset$, stop. Otherwise, proceed to the following steps.

- *Pointing*: For every $i \in I(d-1)$ and i's every pseudo-agent i_h, if h is i's most preferred house, let i_h point to himself. Otherwise, let i_h point to all pseudo-agents $j_{h'}$ such that $j \neq i$ and h' is i's most preferred house in $H(d-1)$. There will be multiple cycles.

- *Selecting Cycles*: We select the following three types of cycles to trade:

  (i) existing-tenant cycles: the cycles consisting only of pseudo-tenants;

  (ii) new-agent self-cycles: the cycles formed by new pseudo-agents pointing to themselves;

  (iii) feasible new-agent cycles: the cycles involving at most two new pseudo-agents and not contained in (i) and (ii).

- *Trading*: For every selected cycle c, the trading quota of c is

$$TQ(c) := \min\{\min_{i \in I(c)} r_i(d-1), \min_{h \in H(c)} \Delta h\},$$

  where $I(c)$ is the set of hosts of the pseudo-agents involved in c, $H(c)$ is the set of houses involved in c, and Δh is the amount of h held by the relevant pseudo-agent involved in c. Then,

  (1) Cycles of types (i) and (ii) are traded immediately with their trading quotas.

  (2) Cycles of type (iii) are traded with a common quota, which equals the smallest trading quota of all type (iii) cycles.

  Any house a pseudo-agent obtains by trading cycles belongs to his host.

- *Leaving*: If a pseudo-agent uses up the house he holds, the pseudo-agent is removed. When an agent i's demand is satisfied, i.e. $r_i(d) = 0$, i leaves the algorithm along with his all pseudo-agents. The remaining endowments of i, if any, are regarded as social endowments and uniformly assigned to the remaining agents in $I(d)$. Go to step $d+1$.

In Appendix C.2 we show how to apply TTC^E to solve the problem in Example 1. Now we prove that PS^E is equivalent to TTC^E.[9]

Theorem 1. *The PS^E mechanism is equivalent to the TTC^E mechanism.*

This theorem implies that PS^E can be equivalently seen as a dynamic process of trading ownerships. In Section 3.3 we have shown that PS^E can be seen as a dynamic process of trading eating speeds. In an eating algorithm agents spend their speeds on the houses they want to consume, and if an agent consumes a house, at the same time he gives up the chances to consume the other houses. So trading eating speeds is equivalent to trading ownerships. Hence Proposition 1 and Theorem 1 actually illustrate the same feature of PS^E.

Although PS is ex ante more efficient than RP, Che and Kojima prove that PS and RP are asymptotically equivalent if the market size properly grows. Using their proof method we similarly prove that PS^E is asymptotically equivalent to random YRMH-IGYT,[10] the generalization of RP to house allocation problems with existing tenants. The intuition behind the result is straightforwardly illustrated by the analogy between "you request my house - I get your speed" and "you request my house - I get your turn". Specifically, the advantage of an existing tenant in PS^E exactly corresponds to the advantage of the same existing tenant in random YRMH-IGYT when the market size is infinitely large.

Theorem 2. *The PS^E mechanism is asymptotically equivalent to the random "you request my house - I get your turn" mechanism.*

Since random YRMH-IGYT is strategy-proof, Theorem 2 implies that PS^E is asymptotically strategy-proof. Liu and Pycia (2013) prove that in house allocation problems all

[9]In our proof we make a slight adjustment of TTC^E such that PS^E is equivalent to TTC^E step by step.

[10]Random YRMH-IGYT proceeds as follows. First randomly draw an ordering of all agents from the uniform distribution. Then let agents sequentially obtain their most preferred objects among remaining ones according to the ordering. But if an agent wants to obtain the private endowment of an existing tenant who has not obtained an object, let the existing tenant points to his most preferred object. If the object is the private endowment of another existing tenant who has not obtained an object, repeat the process until we have a chain or cycle such that all agents in the chain or cycle obtain the objects they want.

mechanisms that are asymptotically ordinally efficient, asymptotically strategy-proof and treating equals equally, must be asymptotically equivalent under some regularity condition. In our model there are existing tenants, so Theorem 2 is not implied by their result. It is unknown that whether their result can be extended to house allocation problems with existing tenants.

6 Comparison with Other Mechanisms

In this section we compare PS^E with the other mechanisms in the literature. Yılmaz (2010) proposes the PS^{IR} mechanism, which is the minimal deviation from PS by satisfying the *individual rationality* (IR) constraint. Specifically, PS^{IR} proceeds by letting agents eat their most preferred houses with the same speed, but if at any time of the procedure the IR constraint of some group of existing tenants binds, PS^{IR} isolates the group and their remaining acceptable houses as a sub-problem by blocking the other agents from consuming those houses. To illustrate it we present the procedure of PS^{IR} in solving Example 1 in Appendix C.1. Yilmaz proves that PS^{IR} is ordinally efficient and satisfies a fairness notion called *no justified-envy* (NJE). Formally, an assignment q satisfies NJE if for any two agents i, j, if q_i is individually rational for j, then i does not envy j. Since the IR constraint of new agents never binds, PS^{IR} must be new-agent envy-free. Hence PS^{IR} has a better fairness property than PS^E. But it is not known that PS^{IR} is immune to the manipulation of any class of strategies. In particular, PS^{IR} is not *boundedly invariant* (see Appendix C.1), and can be manipulated by truncation strategies. So PS^E has a better incentive property than PS^{IR}.

The main difference between PS^E and PS^{IR} lies in their treatments of the ownerships of private endowments. In PS^E existing tenants can trade their ownerships of private endowments with the others, while in PS^{IR} existing tenants cannot. In PS^{IR} existing tenants have advantages over new agents only to the extent that their IR constraint is respected. This difference implies that PS^E and PS^{IR} should be used in different applications. To illustrate it we construct the following example.

Example 3. *There are three agents and two houses. Their preferences and the assignments found by PS^E and PS^{IR} are shown as follows.*

At $t = 0$, i_1, i_3 want to consume h_2 and i_2 wants to consume h_1:

Problem				PS^{IR}				PS^{E}		
$\succsim_1$	$\succsim_2$	$\succsim_3$		i_1	i_2	i_3		i_1	i_2	i_3
h_2	h_1	h_2		$1/2h_2$	$1/2h_1$	$1/2h_2$		$2/3h_2$	$2/3h_1$	$1/3h_2$
$\boxed{h_1}$	h_0	h_0		$1/2h_1$	$1/2h_0$	$1/2h_0$		$1/3h_1$	$1/3h_0$	$2/3h_0$

- *In PS^{IR} all agents have the same eating speed of one. So at $t = 1/2$, h_2 is exhausted. Then to satisfy the IR of i_1, the remaining $1/2h_1$ is exclusively given to i_1.*
- *In PS^{E} agent i_1 has an eating speed of two and others have an eating speed of one. At $t = 1/3$, h_2 is exhausted. Then i_1 consumes $1/3h_1$ by trading a self-cycle. Lastly i_2 consumes $1/3h_1$.*

The two ranodm assignments are implemented by putting different probabilities on two deterministic assignments:

$$PS^{IR} = 1/2\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_0 \end{pmatrix} + 1/2\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_0 & h_2 \end{pmatrix},$$

$$PS^{E} = 2/3\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_0 \end{pmatrix} + 1/3\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_0 & h_2 \end{pmatrix}.$$

If this example is the assignment of on-campus apartments, then the school controls all apartments and can give existing tenants only squatting rights over their current apartments. In this case PS^{IR} is more appropriate since it gives more fairness among agents. In the above example i_1, i_3 have the same chance of obtaining h_2.

If this example is a kidney exchange problem in which h_2 is a non-directional altruistic kidney that is compatible with both i_1 and i_3, while h_1 is the kidney donated to i_1 by his families, but it is incompatible with i_1 yet compatible with i_2. In this case i_1 controls h_1 and can bring h_1 away if he wants to. Since maximizing the number of successful transplants is the main objective in this situation, PS^{E} is more appropriate since with $2/3$ probability there are two transplants and with $1/3$ probability there is only one transplant. In general by giving existing tenants advantages PS^{E} can incentivize them to bring their donated kidneys to the exchange program.

According to the description of Yılmaz (2010), UER of Sethuraman (2001) also trades the cycles between existing tenants as PS^{E} does. But when there are no cycles at any step d, the eating speeds of agents are specified as follows: for $t \geq t_d$, $s_i(t) = 1$ for any

Mechanism	Efficiency	Manipulability	Fairness
PS^E	Ordinally efficient	Weakly dropping-strategy-proof, truncation-strategy-proof, asymptotically strategy-proof	New-agent envy-free
PS^{IR}	Ordinally efficient	None	No justified envy
UER	Ordinally efficient	Weakly dropping-strategy-proof, truncation-strategy-proof	New-agent envy-free
Random YRMH-IGYT	Ex post efficient	Strategy-proof	Weakly new-agent envy-free

Table 3: The properties of multiple mechanisms

$i \in I_N(d-1)$, $s_j(t) = 1$ if $r_j(d-1) < r_{\pi(j)}(d-1)$ and $s_j(t) = s_{\pi(j)}(t)$ if $r_j(d-1) = r_{\pi(j)}(d-1)$ for any $j \in I_E(d-1)$. That is, the eating speed of any existing tenant j becomes equal to the total speed at which his private endowment is being eaten only when his IR constraint is to be violated. But as we said before, there are infinitely many ways to satisfy the IR constraint of existing tenants. We are not aware of any specific reason or any desirable characterization of the speed setting in UER. So it seems that the main motivation behind UER is just to satisfy the IR constraint.[11] In terms of properties UER satisfies those we identify for PS^E in Section 4. However, PS^E is asymptotically strategy-proof, but there is no argument supporting that UER is too.

Random YRMH-IGYT is a desirable mechanism since it is strategy-proof. But random YRMH-IGYT is only ex post efficient and weakly new-agent envy-free. Hence if incentive compatibility is the most important criterion in an application, random YRMH-IGYT outperforms PS^E. But if efficiency or fairness is more important, PS^E is a more desirable mechanism. Since they approximate each other in large markets under some conditions, in some cases there is no big difference between choosing any mechanism. We summarize the properties of the mechanisms we just discussed in Table 3.

[11]In Example 3 UER finds the same assignment as PS^{IR} does.

7 Weak Preferences

In this section we show that the idea of "you request my house - I get your speed" can easily solve the difficulty caused by weak preferences in PS^E. When there are no existing tenants, we essentially propose an adaption of PS to weak preferences.[12] As will be seen, our method is more straightforward to understand and easier to implement than the method of solving maximum network flow problems used by the literature.

We briefly describe our method as follows. We first arbitrarily choose an exogenous ordering $>_H$ of all houses and an exogenous ordering $>_I$ of all agents, which are used to break ties between houses and ties between agents. At any step of PS^E, if an agent i is indifferent between two remaining houses, say h and h', then if $h >_H h'$, we let i point to h and consume it first. If at some time t, h is exhausted but h' still remains, then we let i point to h' and label his consumption of h, denoted by h^i, as "available" for other agents to consume. If h^i is indeed consumed by other agents after time t, we compensate i by letting i consume h' with an additional speed, which equals the speed at which h^i is being consumed. In this way i is exactly compensated without any loss of welfare.

At time t when agent i labels his consumption h^i as available, if another agent k is indifferent between h and a house h'' he consumed earlier, then k must most prefer h among all remaining houses and available consumptions at time t.[13] Then we let k point to i's consumption h^i and label his consumption of h'' as available for other agents to consume. If there exists another agent j who also labels his consumption h^j as available, we let j point to h^i if $i >_I j$. After j's labeling other agents may further label their consumptions as available. So in general at any time t a chain can exist which looks like

$$i_1 \to h_2^{i_2} \to i_2 \to h_3^{i_3} \to i_3 \to \cdots \to h_m^{i_m} \to i_m \to h_{m+1}$$

where every $i_o (o = 2, \ldots, m)$ labels his consumption $h_o^{i_0}$ as available because $h_{o+1} \sim_{i_o} h_o$, h_{m+1} is a remaining house, and i_1 strictly prefers h_2 to all remaining houses and after breaking any possible ties he points to $h_2^{i_2}$. Then there are two cases:

[12] When there are no soical endowments and all agents are existing tenants, PS^E coincides with TTC. So we also essentially propose an adaption of TTC to weak preferences. Our adaption is actually very close to that of Jaramillo and Manjunath (2012).

[13] It is because at any time in the procedure of PS^E agents always consume their most preferred houses. Since k consumed h'' earlier and h'' is as good as h, h must be a best house for k at time t.

- If i_1 eats $h_2^{i_2}$, then every i_o $(o = 2, \ldots, m-1)$ eats $h_{o+1}^{i_{o+1}}$ and i_m eats h_{m+1} with additional speeds as specified in PS^E. But if some agent $i_o(o = 2, \ldots, m)$ in the chain has been full, his eating speed will be equal to the speed at which his consumption $h_o^{i_o}$ is consumed. In this way i_o will never overconsume.

- If i_1 is an existing tenant and h_{m+1} happens to be i_1's private endowment, the above chain becomes a cycle. Then we trade the cycle immediately. But when calculating how much house can be traded, we ignore the residual demands of all $i_o(o = 2, \ldots, m)$ since the trading does not increase their total consumptions.

In the above chain every agent $i_o(o = 1, \ldots, m)$ is eventually linked to a remaining house h_{m+1} through the chain. In general only this kind of chains are *indispensable* to keep ordinal efficiency. Any exchange of available consumptions among agents is unnecessary and may mess our algorithm up. So in our formal definition of the algorithm below, at every step we introduce a pointing stage in which we carefully construct the chains to exclude the possibility that multiple agents point to each other's available consumptions such that they form a cycle.

7.1 Formal Definition

There are three stages at every step of PS^E under weak preferences: *Pointing* stage in which agents point to houses, *Consuming* stage in which agents consume the houses they point to either by eating or trading, and *Labeling* stage in which agents update their sets of available houses. There are an exogenous ordering $>_H$ of all houses and an exogenous ordering $>_I$ of all agents to break all possible ties.

d: step;

$h^0(d)$: The remaining (fractional) house h when step d ends;

$H(d) := \{h^0(d) : h \in H, h^0(d) > 0\}$: set of remaining houses when step d ends;

$I(d)$: set of remaining agents when step d ends;

$h^i(d)$: agent i's consumption of house h when step d ends;

$p_i(d)$: the (fractional) house agent i points to at step d;

$C_i(d) := \{h^i(d) : h \in H, h^i(d) > 0\}$: agent i's consumption profile when step d ends;

$H_i(d)$: set of consumptions that agent i labels as available when step d ends;

Initialization: $H(0) = H$, $I(0) = I$, and $c_i(0) = a_i(0) = \emptyset$ for $\forall i \in I$.

Step $d \geq 1$:

- *Pointing*:

  Define the menu of every $i \in I(d-1)$ as $M_i(d) := H(d-1) \cup H_{-i}(d-1)$, where $H_{-i}(d-1) := \bigcup_{j \in I(d-1): j \neq i} H_j(d-1)$. Then the set of i's most preferred (fractional) houses in $M_i(d)$ is $Ch_i(M_i(d)) := \arg \underset{h \in M_i(d)}{\succsim_i - \max} h$.[14]

  Round 0: For every $i \in I(d-1)$, if the (fractional) house he points to at step $d-1$ is still in $M_i(d)$, let i still point to the house. Formally, if $p_i(d-1) = h^j(d-2)$ and $h^j(d-1) \in M_i(d)$, then $p_i(d) = h^j(d-1)$. Denote the set of all such agents by $P_0(d)$. It is obvious that $P_0(1) = \emptyset$.

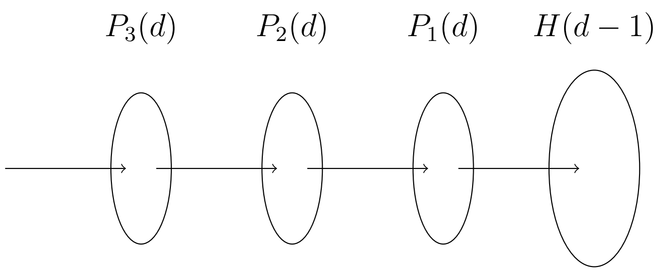


Figure 1: Illustration of the chains we construct

  Round 1: For every $i \in I(d-1) \backslash P_0(d)$, if $Ch_i(M_i(d)) \cap H(d-1) \neq \emptyset$, let i point to the house in $Ch_i(M_i(d)) \cap H(d-1)$ that is ranked highest in $>_H$. Formally,

$$p_i(d) := \underset{h^0(d-1) \in Ch_i(M_i(d)) \cap H(d-1)}{\arg >_H - \max} h^0(d-1)$$

  Note that some agents in $P_0(d)$ may also point to some houses in $H(d-1)$. Then we denote the set of all agents that point to some houses in $H(d-1)$ by $P_1(d)$.

  Round 2: Now for every $i \in I(d-1) \backslash \{P_0(d) \cup P_1(d)\}$ we know that $Ch_i(M_i(d)) \subset H_{-i}(d-1)$. Then for every i who most prefers some available consumption held by

[14]Note that $M_i(d)$ may contain more than one fractions of the same house, for example, $h^0(d-1)$ and $h^j(d-1)$. Then i is indifferent between them.

$P_1(d)$, that is, $Ch_i(M_i(d)) \cap (\cup_{j \in P_1(d)} H_j(d-1)) \neq \emptyset$, we let

$$p_i(d) := \underset{h^j(d-1) \in Ch_i(M_i(d)) \cap a_j(d-1)}{\arg >_H - \max} h^j(d-1), \tag{1}$$

$$\text{where } j := \underset{j' \in J_i(d)}{\arg >_I - \max} j', \tag{2}$$

$$\text{where } J_i(d) := \underset{j'' \in P_1(d) : Ch_i(M_i(d)) \cap H_{j''}(d-1) \neq \emptyset}{\arg >_H - \max} p_{j''}(d). \tag{3}$$

That is, among the set of agents in $P_1(d)$ who hold i's most preferred available consumption, $J_i(d)$ are those who point to the house in $H(d-1)$ that is ranked highest in $>_H$. Then among $J_i(d)$ we choose the agent j who is ranked highest in $>_I$. Finally, among i's most preferred available consumptions held by j, i points to the consumption that is ranked highest in $>_H$.

We denote the set of the agents discussed in this around, and those in $P_0(d)$ who also point to some available consumption held by $P_1(d)$, by $P_2(d)$.

Round 3: For every $i \in I(d-1) \backslash \{P_0(d) \cup P_1(d) \cup P_2(d)\}$ who most prefers some available consumption held by $P_2(d)$, that is, $Ch_i(M_i(d)) \cap (\cup_{j \in P_2(d)} H_j(d-1)) \neq \emptyset$, we let

$$p_i(d) := \underset{h^j(d-1) \in Ch_i(M_i(d)) \cap H_j(d-1)}{\arg >_H - \max} h^j(d-1), \tag{4}$$

$$\text{where } j := \underset{j' \in J_i(d)}{\arg >_I - \max} j', \tag{5}$$

$$\text{where } J_i(d) := \underset{j'' \in P_2(d) : Ch_i(M_i(d)) \cap H_{j''}(d-1) \neq \emptyset}{\arg >_H - \max} p^2_{j''}(d). \tag{6}$$

Here $p^2_{j''}(d)$ is the house that the owner of $p_{j''}(d)$ points to at step d. In other words, it is the house in $H(d-1)$ to which $j'' \in P_2(d)$ is linked through the chains we construct. Then the explanation of (4)-(6) is that, among the set of agents in $P_2(d)$ who hold i's most preferred available consumption, $J_i(d)$ are those who are linked to a house in $H(d-1)$ that is ranked highest in $>_H$. Then among $J_i(d)$ we choose the agent j that is ranked highest in $>_I$. Finally, among i's most preferred available consumptions held by j, i points to the consumption that is ranked highest in $>_H$.

We denote the set of the agents discussed in this around, and those in $P_0(d)$ who also point to some available consumption held by $P_2(d)$, by $P_3(d)$.

...

Round n: For every $i \in I(d-1)\setminus \cup_{x=1}^{n-1} P_x(d)$ who most prefers some available consumption held by $P_{n-1}(d)$, that is, $Ch_i(M_i(d)) \cap (\cup_{j\in P_{n-1}(d)} H_j(d-1)) \neq \emptyset$, we let

$$p_i(d) := \underset{h^j(d-1)\in Ch_i(M_i(d))\cap H_j(d-1)}{\arg >_H - \max} h^j(d-1), \tag{7}$$

$$\text{where } j := \underset{j'\in J_i(d)}{\arg >_I - \max}\, j', \tag{8}$$

$$\text{where } J_i(d) := \underset{j''\in P_{n-1}(d): Ch_i(M_i(d))\cap H_{j''}(d-1)\neq\emptyset}{\arg >_H - \max} p_{j''}^{n-1}(d) \tag{9}$$

Here $p_{j''}^{n-1}(d)$ is the house in $H(d-1)$ to which $j'' \in P_{n-1}(d)$ is linked through the chains we construct. Then the explanation of (7)-(9) is that, among the set of agents in $P_{n-1}(d)$ who hold i's most preferred available consumption, $J_i(d)$ are those who are linked to a house in $H(d-1)$ that is ranked highest in $>_H$. Then among $J_i(d)$ we choose the agent j that is ranked highest in $>_I$. Finally, among i's most preferred available consumptions held by j, i points to the consumption that is ranked highest in $>_H$.

We denote the set of the agents discussed in this around, and those in $P_0(d)$ who also point to some available consumption held by $P_{n-1}(d)$, by $P_n(d)$.

Since there are finite agents, the above procedure must stop in some finite rounds. Then every agent in $I(d-1)$ points to some house. Then we let every private endowment and available consumption point to its owner.

- *Consuming*:

  Run the consuming stage of PS^E with the following two remarks:

  - An agent whose demand has been satisfied can consume again only if his available consumptions are consumed by other agents. His eating speed equals the total speed at which his consumptions are consumed.
  - If an agent is involved in a cycle along with his available consumption, his residual demand is omitted in calculating the trading quota of the cycle.

This stage ends if an available consumption or a remaining house is exhausted, or a cycle is traded, or an agent's demand is satisfied. Let $C_i(d)$ be the new consumption profile of every $i \in I(d-1)$ and $H(d)$ be the new set of remaining houses. If $H(d) = \emptyset$, stop the algorithm.

- *Labeling*:

We update the available consumption sets of agents in $I(d-1)$ sequentially in the following rounds.

Round 1: For every $i \in I(d-1)$, the available consumption set of i is $H_i(d) := \{h^i(d) : h^i(d) \in C_i(d) \text{ and } h \sim_i \hat{h} \text{ for some } \hat{h}^0(d) \in H(d)\}$. Denote the set of such agents by $L_1(d)$.

Round 2: For every $i \in I(d-1)\backslash L_1(d)$, the available consumption set of i is $H_i(d) := \{h^i(d) : h^i(d) \in C_i(d) \text{ and } h \sim_i \hat{h} \text{ for some } \hat{h}^k(d) \in \bigcup_{j \in L_1(d)} H_j(d)\}$. Denote the set of such agents by $L_2(d)$.

...

Round n: For every agent $i \in I(d-1)\backslash\{L_1(d) \cup L_2(d) \cup \cdots \cup L_{n-1}(d)\}$, the available consumption set of i is $H_i(d) := \{h^i(d) : h^i(d) \in C_i(d) \text{ and } h \sim_i \hat{h} \text{ for some } \hat{h}^k(d) \in \bigcup_{j \in L_{n-1}(d)} H_j(d)\}$. Denote the set of such agents by $L_n(d)$.

Since there are finite agents, the above process must finish in finite rounds. Then any agent who is full and has an empty set of available consumptions leaves the algorithm with his consumption. Then the set of remaining agents is denoted by $I(d)$. If $I(d) = \emptyset$, stop the algorithm. Otherwise, go to step $d+1$.

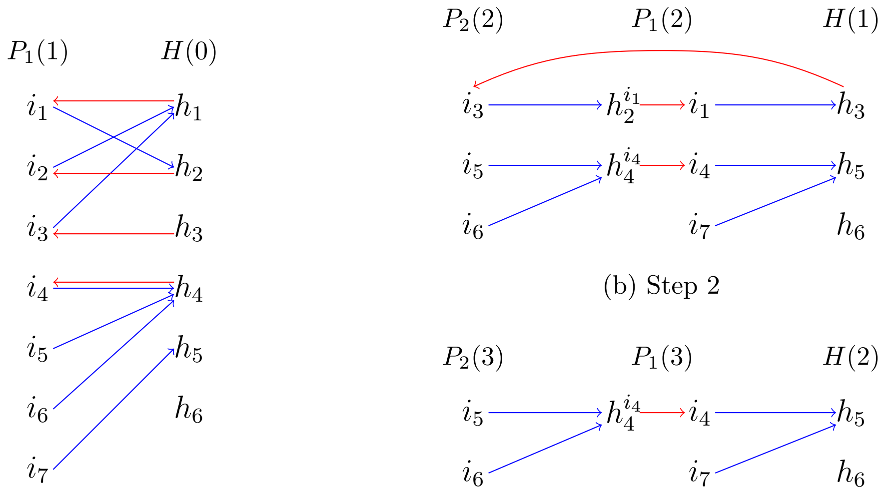
7.2 An Example

Example 4. *A problem consists of $H = \{h_0, h_1, \ldots, h_6\}$ and $I = \{i_1, i_2, \ldots, i_7\}$. i_1, i_2, i_3, i_4 are existing tenants and own h_1, h_2, h_3, h_4 respectively. i_5, i_6, i_7 are new agents and h_5, h_6 are vacant houses. The following table is the preference profile of all agents where $\succsim_o$ is i_o's preference list. Unacceptable houses for existing tenants are omitted from their preference lists.*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$	$\succsim_7$
h_2, h_3	h_1	h_1	$\boxed{h_4}, h_5$	h_4, h_5	h_4, h_6	h_5
$\boxed{h_1}$	h_3	h_2		h_6	h_5	h_6
	$\boxed{h_2}$	$\boxed{h_3}$		h_0	h_0	h_0

The two exogenous orderings are: $i_1 >_I i_2 >_I i_3 >_I i_4 >_I i_5 >_I i_6$ and $h_0 >_H h_1 >_H h_2 >_H h_3 >_H h_4 >_H h_5 >_H h_6$.

Step 1: *The pointing stage is shown as the following graph. There are two cycles: $i_1 \to h_2 \to i_2 \to h_1 \to i_1$ and $i_4 \to h_4 \to i_4$. After trading these cycles, i_1 gets h_2 and labels it as available since $h_2 \sim_1 h_3$, i_2 gets h_1 and leaves the algorithm, and i_4 gets h_4 and labels it as available since $h_4 \sim_4 h_5$.*



(a) Step 1

(b) Step 2

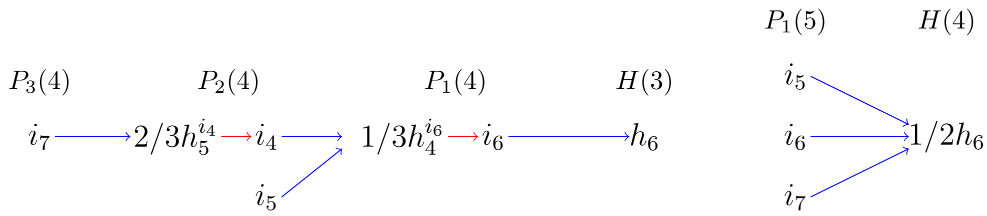
(c) Step 3

Figure 2: Steps 1, 2 and 3

Step 2: *The pointing stage is shown as the following graph. In particular, i_5, i_6, i_7 point to the same houses as they did at step 1. There is one cycle: $i_1 \to h_3 \to i_3 \to h_2^{i_1} \to i_1$. After trading the cycle i_3 gets h_2 and leaves the algorithm. Then i_1 gets h_3 and also leaves the algorithm. i_4 still labels his consumption of h_4 as available since $h_4 \sim_4 h_5$.*

Step 3: *i_4, i_5, i_6, i_7 point to the same houses as they did in Step 2. Since i_4 is full, i_4's eating speed is two. Every other agent' eating speed is one. So h_5 is exhausted at*

$t = 1/3$, and i_4 gets $2/3h_5$, i_5, i_6 each get $1/3h_4$, and i_7 gets $1/3h_5$. In round 1 of the labeling stage i_6 labels his $1/3h_4$ as available since $h_4 \sim_6 h_6$. Then in round 2 both i_4 and i_5 label all of their consumption as available.



(a) Step 4 (b) Step 5

Figure 3: Steps 4 and 5

Step 4: *The pointing stage is shown as the following graph. Since i_4 is full, his eating speed is one. Thus i_6's eating speed is three. Every other agent's eating speed is one. At $t = 1/2$, i_6's $1/3h_4$ is exhausted. Then in the labeling stage only i_6 labels his $1/2h_6$ as available. Since i_4 is full, he leaves the algorithm.*

Step 5: *All of i_5, i_6, i_7 point to h_6. At $t = 2/3$, h_6 is exhausted. Stop the algorithm.*

The final assignment is shown as follows:

i_1	i_2	i_3	i_4	i_5	i_6	i_7
h_3	h_1	h_2	$1/2h_4$	$1/2h_4$	$2/3h_6$	$1/2h_5$
			$1/2h_5$	$1/6h_6$		$1/6h_6$

7.3 Properties

We prove that the above adaption of PS^E to weak preferences has the same properties as before.

Proposition 4. *The PS^E mechanism under weak preferences is ordinally efficient, individually rational, new-agent envy-free, boundedly invariant, weakly dropping-strategy-proof, and truncation-strategy-proof.*

Depending on how weak preferences are resolved in TTC^E and Random YRMH-IGYT, the equivalence theorems proved before may not hold anymore. That is, an agent may obtain different assignments in PS^E and TTC^E (or random YRMH-IGYT). However, we conjecture that there is some way to keep the theorems hold by proving that the welfare of the same agent is still (asymptotically) same in the two mechanisms.

8 Conclusion

This paper proposes a new mechanism to solve random assignment problems in which some agents have private endowments. The new mechanism PS^E generalizes PS by letting agents benefit from the popularity of their private endowments, which is illustrated by the idea of "you request my house - I get your speed". Interestingly, the same idea can also be used to deal with weak preferences in a simple way.

Now we discuss two directions for future research. Athanassoglou and Sethuraman (2011) introduce a problem in which agents have fractional endowments. They generalize PS^{IR} to the problem and prove that it satisfies NJE. However, since two agents of equal endowments bring exactly the same resources to the problem, they also believe that *equal-endowment no envy (EENE)* is a more reasonable fairness criterion than NJE. But their generalization of PS^{IR} does not satisfy EENE, and they are not aware of any mechanism satisfying ordinal efficiency, individual rationality and EENE. We believe that PS^E can be extended to their problem. In particular, if two agents have equal endowments, in the mechanism they should always have equal eating speeds or be involved in equivalent trading cycles. Then EENE will be satisfied. In another direction, since Kesten and Ünver (2015) use a different mechanism than PS to solve random assignment problems with priorities, it is interesting to check whether PS can be properly adapted to solve the problems.

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A Proofs of Theorem 1 and Theorem 2

A.1 Proof of Theorem 1

To prove Theorem 1 we first prove a lemma about PS^E and two lemmas about TTC^E.

A lemma about PS^E:

For any $d \geq 0$, at the beginning of step $d+1$ in PS^E, if there are no cycles among existing tenants, since every agent in $I(d)$ points to a house and every private endowment in $H(d)$ points to its owner, every agent must be directly linked to a house and may be further indirectly linked to other agents and houses through some paths (see Figure 4). In particular, every agent must be linked to a social endowment, which is either a vacant house or a private endowment whose owner has stopped consuming, through a unique path. Then we prove that for any $h \in H(d)$, $s_h(t_d)$ is equal to the number of agents who are linked to h through some paths (see h_2, h_3 in Figure 4), and $s_i(t_d)$ is equal to the number of agents linked to i by including i himself.

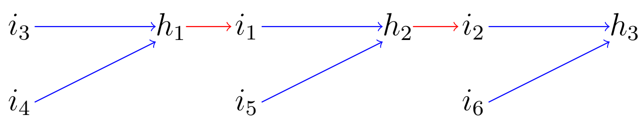


Figure 4: Illustration of Lemma 1. h_1, h_2 are private endowments of i_1, i_2, and h_3 is a social endowment. It is easy to see that $s_{h_1}(t) = 2$, $s_{i_1}(t) = 3$, $s_{h_2}(t) = 4$, $s_{i_2}(t) = 5$, and $s_{h_3}(t) = 6$.

Lemma 1. *For all $h \in H(d)$ and all $i \in I(d)$, $s_h(t_d)$ is equal to the number of agents who are linked to h, and $s_i(t_d)$ is equal to the number of agents linked to i by including i himself.*

Proof. For all house h in $H(d)$, recall that $A_h(t_d)$ is the set of agents who point to h and $s_h(t_d) = \sum_{i \in A_h(t_d)} s_i(t_d)$.

For all h such that all agents in $A_h(t_d)$ have an eating speed of one, it is obvious that $s_h(t) = |A_h(t_d)|$. On the hand other, $A_h(t_d)$ is also the set of all agents linked to h. If h is a private endowment of some i, then it is obvious that $s_i(t_d)$ is equal to the number of agents in $A_h(t_d)$ added by one. Denote the set of such existing tenants by $X_1(d)$.

For all h such that all existing tenants in $A_h(t_d)$ are in $X_1(d)$, given the above result it is obvious that $s_h(t_d)$ is equal to the number of all agents in $A_h(t_d)$ added by the number of other agents linked to h through existing tenants in $A_h(t_d)$. Denote the set of existing tenants whose private endowments belong to this case by $X_2(d)$.

...

By induction, $s_h(t_d)$ is equal to the number of agents who are linked to h. Then it is obvious that $s_i(t_d)$ is equal to the number of agents linked to i by including i. □

With some abuse of notations we still use $s_h(t_d)$ to denote the set of agents linked to h. Since every agent is linked to a unique vacant house, all agents in $I(d)$ can be partitioned into $\{s_h(t_d)\}_{h \in H_V(d)}$, where $H_V(d)$ is the set of social endowments. So $\cup_{h \in H_V(d)} s_h(t_d)$ $= I(d)$ and $\sum_{h \in H_V(d)} s_h(t_d) = |I(d)|$. We use $H_O(d) := H(d) \backslash H_V(d)$ to denote the set of private endowments.

Two lemmas about TTC^E:

In TTC^E, for any $d \geq 0$ we define $B_h(d)$ and $w_h(d)$ as the counterparts of $A_h(t_d)$ and $s_h(t_d)$. That is, $B_h(d)$ is the set of agents whose most preferred remaining house is h, and $w_h(d)$ is both the number and the set of agents linked to h through some paths in the graph of TTC^E (see Appendix C.2). We still use $H_V(d), H_O(d)$ to denote the set of social endowments and the set of private endowments. Then $\sum_{h \in H_V(d)} w_h(d) = |I(d)|$ and $\cup_{h \in H_V(d)} w_h(d) = I(d)$.

At the beginning of step one, if two pseudo-agents hold a same house, the house must be a social endowment and they must hold an equal fraction of it. In the proof of Theorem 1 we will show that this statement is still true in the procedure of TTC^E. In the following we prove two lemmas about TTC^E.

Lemma 2. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, every new pseudo-agent i_h where $i \in I(d)\backslash B_h(d)$ and $h \in H_V(d)$ is involved in a total number of $w_h(d)$ of selected cycles.*

Proof. All remaining agents in $I(d)$ can be partitioned into $B_h(d)$ and $I(d)\backslash B_h(d)$. $B_h(d)$ can be further partitioned into $B_{Nh}(d)$ and $B_{Eh}(d)$, the set of new agents in $B_h(d)$ and the set of existing tenants in $B_h(d)$. For every existing tenant $j \in B_{Eh}(d)$, there are $w_{\pi(j)}(d)$ agents who are linked to $\pi(j)$ and all of them are further linked to h. So by the definition of $w_h(d)$, $\sum_{j \in B_{Eh}(d)} w_{\pi(j)}(d) + |B_h(d)| = w_h(d)$. For any new pseduo-agent i_h such that $i \in I(d)\backslash B_h(d)$ and $h \in H_V(d)$, i's most preferred remaining house is not h. There are two cases:

- If i's most preferred remaining house is another $h' \in H_V(d)$, then i_h must point to all pseudo-agents $\{a_{h'}\}_{a \in I(d)\backslash\{i\}}$. Then for every $k_{h'}$ such that $k \in B_h(d)$, i_h and $k_{h'}$ point to each other and form a cycle of length two. So i_h is involved in a total number of $|B_h(d)|$ of such length-two cycles. All these cycle are selected to trade. Moreover, since for every $j \in B_{Eh}(d)$ there are $w_{\pi(j)}(d)$ agents linked to j, correspondingly there are $w_{\pi(j)}(d)$ new pseudo-agents who hold h' and are linked to $j_{\pi(j)}$. Since i_h points to all the $w_{\pi(j)}(d)$ new pseudo-agents and $j_{\pi(j)}$ points to i_h, there are $w_{\pi(j)}(d)$ cycles of the form $i_h \to \ell_{h'} \to \cdots \to j_{\pi(j)} \to i_h$ where $\ell_{h'}$ is a typical new pseudo-agent linked to $j_{\pi(j)}$. Note that i_h and $\ell_{h'}$ are the only two new pseudo-agents in the cycle. So all these cycles are selected to trade. Hence the total number of selected cycles that involve i_h is $\sum_{j \in B_{Eh}(d)} w_{\pi(j)}(d) + |B_h(d)| = w_h(d)$.

- If i's most preferred remaining house is a private endowment $h' \in H_O(d)$, then i_h points only to $g_{h'}$ where g is the owner of h'.

  If g's most preferred remaining house is some $h'' \in H_V(d)$ other than h, then g points to all pseudo-agents $\{a_{h''}\}_{a \in I(d)\backslash g}$. Similarly as in the first case, for every $k_{h''}$ such that $k \in B_h(d)$ there is a length-three cycle $i_h \to g_{h'} \to k_{h''} \to i_h$ that involves only two new pseudo-agents. All these cycles are selected to trade and their number is $|B_h(d)|$. Moreover, for every $j \in B_{Eh}(d)$, there are $w_{\pi(j)}(d)$ new pseudo-agents who hold h'' and are linked to $j_{\pi(j)}$. For every such new pseudo-agent $\ell_{h''}$ there is a cycle $i_h \to g_{h'} \to \ell_{h''} \to \cdots \to j_{\pi(j)} \to i_h$ that involves only two new pseudo-agents

i_h and $\ell_{h''}$. So all these cycles are selected to trade. Hence the total number of selected cycles that involve i_h is $\sum_{j \in B_{Eh}(d)} w_{\pi(j)}(d) + |B_h(d)| = w_h(d)$.

If g's most preferred remaining house is h, then i_h is involved in only one cycle: $i_h \to g_{h'} \to i_h$ in which i_h is the only new pseudo-agent. Then we regard the cycle as $w_h(d)$ cycles.

If g's most preferred remaining house is in $H_O(d)$, then g must be eventually linked to a house in $H_V(d)$. If the house is not h, then as proved before i_h is involved in $w_h(d)$ selected cycles. If the house is h, then i_h is involved in only one cycle looking like $i_h \to g_{h'} \to \cdots \to j'_{\pi(j')} \to i_h$ in which i_h is the only new pseudo-agent. Then we also regard the cycle as $w_h(d)$ cycles.

□

In the above proof when i_h is the only new pseudo-agent in a cycle, we regard the cycle as $w_h(d)$ cycles. Now if a new pseudo-agent i_h most prefers h and points to himself, we also regard the self-cycle as $w_h(d)$ cycles. In our definition of PS^E these cycles may be traded in only one step. But after this trick they are traded in several steps. This trick does not change the final assignment of TTC^E, but it can simplify the proof of Theorem 1. Then by Lemma 2 the set of new pseudo-agents $\{i_h\}_{h \in H_V(d)}$ are totally involved in $\sum_{h \in H_V(d)} w_h(d) = |I(d)|$ selected cycles. This implies the following corollary.

Corollary 1. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, then every new agent $i \in I_N(d)$ is involved in $|I(d)|$ selected cycles.*

However, every existing tenant is further represented by a pseudo-tenant.

Lemma 3. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, then every pseudo-tenant $i_{\pi(i)}$ where $i \in I_E(d)$ is involved in $|I(d)| \cdot w_{\pi(i)}(d)$ selected cycles.*

Proof. For any $i \in I_E(d)$, if i's most preferred remaining house is some $h \in H_V(d)$, then $i_{\pi(i)}$ points to all pseudo-agents $\{a_h\}_{a \in I(d) \setminus \{i\}}$. For every agent $j \in w_{\pi(i)}(d)$, j_h is linked to $i_{\pi(i)}$. So there is a cycle $i_{\pi(i)} \to j_h \to \cdots \to i_{\pi(i)}$ which involves only one new pseudo-agent j_h and it is regarded as $w_h(d)$ selected cycles. Every new pseudo-agent $j_{h'}$ such that $h' \in H_V(d) \setminus \{h\}$ is also linked to $i_{\pi(i)}$. By Lemma 2, $j_{h'}$ is involved in $w_{h'}(d)$ selected cycles. So all these selected cycles also involve $i_{\pi(i)}$. By Corollary 1 we know that for every $j \in w_{\pi(i)}(d)$, $\{j_{h'}\}_{h' \in H_V(d)}$ is totally involved in $|I(d)|$ selected cycles. So $i_{\pi(i)}$ is totally involved in $|I(d)| \cdot w_{\pi(i)}(d)$ selected cycles.

If i's most preferred remaining house is in $H_O(d)$, then $i_{\pi(i)}$ must be linked to a pseudo-tenant whose most preferred remaining house is in $H_V(d)$. Then a similar argument as above proves that each of the $|I(d)|$ selected cycles that involves every $j \in w_{\pi(i)}(d)$ must also involve $i_{\pi(i)}$. So $i_{\pi(i)}$ is totally involved in $|I(d)| \cdot w_{\pi(i)}(d)$ selected cycles. □

Now we know that every existing tenant $i \in I_E(d)$ is totally involved in $|I(d)| + |I(d)| \cdot w_{\pi(i)}(d) = |I(d)| \cdot [w_{\pi(i)}(d) + 1]$ selected cycles.

Corollary 2. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, then every existing tenant $i \in I_E(d)$ is involved in $|I(d)| \cdot [w_{\pi(i)}(d) + 1]$ selected cycles.*

Proof of Theorem 1:

Recall that in Lemmas 2 and 3 we use the *trick* that each cycle in TTC^E that only involves one new pseudo-agent i_h at step $d+1$ is regarded as $w_h(d)$ cycles. Now we use another *trick* that if there are existing-tenants cycles at any step $d+1$ of TTC^E, we trade existing-tenants cycles at step $d+1$, and trade the remaining cycles at step $d+2$. Both tricks do not change the outcome of TTC^E, but they will make TTC^E equivalent to PS^E step by step and hence simplify our proof.

1. We prove the equivalence statement for step $d = 1$.

   At the beginning of step one, all agents and their corresponding pseudo-agents point to same houses in both mechanisms. So $A_h(t_0) = B_h(0)$ and $s_h(t_0) = w_h(0)$ for all $h \in H(0)$. There are two cases to consider.

   1.a If there are existing-tenant cycles in PS^E, then there must be same existing-tenant cycles in TTC^E. Denote the set of these cycles by $C(1)$. Both mechanisms trade them immediately. Let $TQ^P_c(d)$ and $TQ^T_c(d)$ be the trading quotas of cycle c at step d of PS^E and TTC^E respectively, then it is obvious that $TQ^P_c(1) = TQ^T_c(1)$ for every cycle $c \in C(1)$. Let $TQ^P_h(d)$ and $TQ^T_h(d)$ be the fraction of house h that are traded at step d of PS^E and TTC^E respectively, then $TQ^P_h(1) = TQ^T_h(1)$ for every house $h \in H_O(0)$. So the two mechanisms are equivalent at step one.

1.b If there are no existing-tenant cycles in PS^E, step one of PS^E must end with a house being exhausted. Denote the house by $h^P(1)$, then $s_{h^P(1)}(t_0) \geq s_h(t_0)$ for any $h \in H(0)$. Let $EPS(d)$ (eating per speed) be the fraction of houses an agent with speed one eats at step d, then $EPS(1) = 1/s_{h^P(1)}(t_0)$. So every new agent $i \in I_N(0)$ eats $EPS(1)$ of his most preferred house, and every existing tenant $j \in I_E(0)$ eats $s_j(t_0) \cdot EPS(1)$ of his most preferred house.

In TTC^E, by Lemmas 2 and 3 every new pseudo-agent i_h holds $1/|I(0)|$ of $h \in H_V(0)$ and is involved in $w_h(0)$ selected cycles, and every pseudo-tenant $i_{h'}$ holds $h' \in H_O(0)$ and is involved in $|I(0)| \cdot w_{\pi(i)}(0)$ selected cycles. Since all selected cycles are traded with a common quota, step one of TTC^E must end with a house being exhausted and the house solves the following problem:

$$\min_h \{ \min_{h \in H_V(0)} \frac{1/|I(0)|}{w_h(0)}, \min_{h \in H_O(0)} \frac{1}{|I(0)| w_h(0)} \} \Leftrightarrow \min_{h \in H(0)} \frac{1}{|I(0)| w_h(0)}.$$

Since $s_h(t_0) = w_h(0)$, the solution to the above problem must be $h^P(1)$. Let $TPC(d)$ (trading per cycle) be the common trading quota at step d of TTC^E. Then $TPC(1) = 1/[|I(0)| w_{h^P(1)}(0)] = 1/[|I(0)| s_{h^P(1)}(t_0)]$. So $|I(0)| TPC(1) = EPS(1)$. Hence every new agent $i \in I_N(0)$ obtains $|I(0)| TPC(1) = EPS(1)$ of his most preferred house, and every existing tenant $j \in I_E(0)$ obtains $|I(0)|[w_{\pi(j)}(0) + 1] TPC(1) = s_j(t_0) EPS(1)$ of his most preferred house. So the two mechanisms are equivalent at step one.

2. Suppose at every step $d \leq k$ the two mechanisms are equivalent. In particular, $|I(d-1)| TPC(d) = EPS(d)$, $TQ_h^P(d) = TQ_h^T(d)$ for every $h \in H_O(d-1)$, and $s_{h'}(t_{d-1}) = w_{h'}(d-1)$ for every $h' \in H(d-1)$. Then we prove that the two mechanisms are still equivalent at step $k+1$. There are three cases to consider.

2.a If there are existing-tenant cycles in PS^E, then since PS^E is equivalent to TTC^E in previous steps, the same cycles must exist at step $k+1$ of TTC^E. Denote the set of cycles by $C(k+1)$. Both mechanisms trade these cycles immediately. Since every agent has the same residual demand and every house has the same remainder in both mechanisms, $TQ_c^P(k+1) = TQ_c^T(k+1)$ for every $c \in C(k+1)$ and $TQ_h^P(k+1) = TQ_h^T(k+1)$ for every $h \in H_O(k)$. So every agent i in $I(k)$ must obtain the same fraction of the same house in both mechanisms, and the two mechanisms are equivalent at step $k+1$.

2.b If there are no existing-tenant cycles and step $k+1$ of PS^E ends with some house $h^P(k+1)$ being exhausted, then we divide the previous k steps into two sets: the set $\alpha(k)$ of steps at which there are existing-tenant cycles and the set $\beta(k)$ of steps at which there are no existing-tenant cycles. By definition $h^P(k+1)$ is a solution to the following problem:

$$EPS(k+1) = \min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ_h^P(d) - \sum_{d\in\beta(k)} s_h(t_{d-1})EPS(d)}{s_h(t_k)}, \quad (10)$$

and $EPS(k+1) \leq \frac{r_i^P(k)}{s_i(t_k)}$ for every $i \in I(k)$ where $r_i^P(k)$ is agent i's residual demand.

In TTC^E every pseudo-agent i_h points to the same house as i does in PS^E. We prove that step $k+1$ of TTC^E must end with $h^P(k+1)$ being exhausted. Since PS^E and TTC^E are equivalent step by step before step $k+1$, the previous k steps of TTC^E can also be partitioned into $\alpha(k)$ and $\beta(k)$. So every remaining agent in $I(k)$ holds an equal fraction of $[1-\sum_{d\in\alpha(k)} TQ_h^T(d) - \sum_{d\in\beta(k)} |I(d-1)|w_h(d-1)TPC(d)]/|I(k)|$ of every $h \in H_V(k)$.[15]

By induction assumption $|I(d-1)|TPC(d) = EPS(d)$, $TQ_h^P(d) = TQ_h^T(d)$ for every $h \in H_O(d-1)$, and $s_{h'}(t_{d-1}) = w_{h'}(d-1)$ for every $h' \in H(d-1)$, equation (10) implies that $h^P(k+1)$ is also a solution to the following problem:

$$TPC(k+1) = \min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ_h^T(d) - \sum_{d\in\beta(k)} |I(d-1)|w_h(d-1)TPC(d)}{|I(k)|w_h(k)}, \quad (11)$$

and $TPC(k+1) \leq \frac{r_i^T(k)}{|I(k)|[w_{\pi(i)}(k)+1]}$ for every $i \in I(k)$ where $r_i^T(k)$ is agent i's residual demand, since $r_i^T(k) = r_i^P(k)$.

So step $k+1$ of TTC^E ends with $h^P(k+1)$ being exhausted and $|I(k)|TPC(k+1) = EPS(k+1)$. Hence every new agent $i \in I_N(k)$ obtains $|I(k)|TPC(k+1) = EPS(k+1)$ of his most preferred house, and every existing tenant $j \in I_E(k)$ obtains $|I(k)|[w_{\pi(j)}(k)+1]TPC(k+1) = s_j(t_k)EPS(k+1)$ of his most preferred house. So the two mechanisms are equivalent at step $k+1$.

[15]Recall that when an agent leaves the algorithm, his remaining endowments are uniformly assigned to remaining agents.

2.c If there are no existing-tenant cycles and step $k+1$ of PS^E ends with some existing tenant, denoted by $i^P(k+1)$, being full and leaving the algorithm, as before we divide the previous k steps into $\alpha(k)$ and $\beta(k)$. Then by the definition of $i^P(k+1)$ we have

$$\begin{aligned} EPS(k+1) &= \frac{r^P_{i^P(k+1)}(k)}{s_{i^P(k+1)}(t_k)} \\ &\leq \min\{\min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ^P_h(d) - \sum_{d\in\beta(k)} s_h(t_{d-1})EPS(d)}{s_h(t_k)}, \min_{i\in I(k)} \frac{r^P_i(k)}{s_i(t_k)}\} \end{aligned}$$

where $r^P_i(k)$ is agent i's residual demand.

Now in TTC^E by the induction assumption we have

$$\begin{aligned} TPC(k+1) &= \frac{r^T_{i^P(k+1)}(k)}{|I(k)|[w_{\pi(i^P(k+1))}(k)+1]} \\ &\leq \min\{\min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ^T_h(d) - \sum_{d\in\beta(k)} |I(d-1)|w_h(d-1)TPC(d)}{|I(k)|w_h(k)}, \\ &\quad \min_{i\in I(k)} \frac{r^T_i(k)}{|I(k)|[w_{\pi(i)}(k)+1]}\}, \end{aligned}$$

where $r^T_i(k)$ is agent i's residual demand and $r^T_i(k) = r^P_i(k)$.

This implies that step $k+1$ of TTC^E also ends with $i^P(k+1)$ being full and leaving the algorithm. Moreover, $|I(k)|TPC(k+1) = EPS(k+1)$. So as before the two mechanisms are still equivalent at step $k+1$.

3. So by induction PS^E is equivalent to TTC^E at every step.

A.2 Proof of Theorem 2

Our proof is based on Che and Kojima (2010). Specifically, we first define finite problems and their limit. Then we characterize the procedure of PS^E in these problems by a few parameters, and prove that PS^E is equivalent to random YRMH-IGYT in the limit problem. In the on-line appendix we prove the convergence of the two mechanisms in finite problems to the limit problem. We omit detailed explanations of the technique here since it has been explained very well by Che and Kojima.

Given an initial problem $m = \{I, H, \pi, \succsim_I\}$, for every $\ell \in \mathbb{N}\backslash\{0\}$ an ℓ-problem is $m^\ell = (I^\ell, H^\ell, \pi^\ell, (\gamma_i)_{i\in I^\ell})$ in which every non-null house in H and every existing tenant in

I have ℓ copies along with his private endowment.[16] γ_i is agent i's preference type, which is a one-to-one mapping $\gamma_i : H \to \{1, ..., n+1\}$ where $n+1 = |H|$ such that $\gamma_i(h) < \gamma_i(h')$ if and only if $h \succ_i h'$. The set of all preference types is denoted by Γ. We partition the set of new agents I_N^ℓ into $\{I_{N\gamma}^\ell\}_{\gamma \in \Gamma}$ where $I_{N\gamma}^\ell$ is the set of new agents with preference type γ. Let $a_{N\gamma}^\ell := \frac{|I_{N\gamma}^\ell|}{\ell}$ be the per-unit number of new agents with preference type γ. The set of new agents can grow in different ways in the size of the problem. However, we require that there exists $a_{N\gamma}^\infty \in \mathbb{R}_+$ such that $a_{N\gamma}^\ell \to a_{N\gamma}^\infty$ for every preference type γ as $\ell \to +\infty$. We define $\{I_{E\gamma}^\ell\}_{\gamma \in \Gamma}$ similarly and it is obvious that $\frac{|I_{E\gamma}^\ell|}{\ell} = |I_{E\gamma}^1|$ for any $\ell \in \mathbb{N} \cup \{\infty\}$ and γ.

Now we characterize the procedure of PS^E in an ℓ-problem by a few parameters. For any set of houses $H' \subseteq H$, let $Ch_\gamma(H')$ be the set of houses in H' that most preferred by the preference type $\gamma \in \Gamma$. When the set of remaining agent is $I^{\ell\prime}$ and the set of remaining houses is $H^{\ell\prime}$, let $S_i^\ell(H^{\ell\prime}, I^{\ell\prime}) := S_{\pi^\ell(i)}^\ell(H^{\ell\prime}, I^{\ell\prime}) + 1$ be the per-unit eating speed of agent i, and let $S_h^\ell(H^{\ell\prime}, I^{\ell\prime}) := \sum_{\gamma \in \Gamma : h \in Ch_\gamma(H')} \{a_{N\gamma}^\ell + \sum_{i \in I_{E\gamma}^{\ell\prime}} S_i^\ell(H^{\ell\prime}, I^{\ell\prime})\}$ be the per-unit speed at which house $h \in H^{\ell\prime}$ is being eaten. Then given $H^\ell(d-1)$, $I^\ell(d-1)$, t_{d-1}^ℓ, $\{r_h^\ell(d-1)\}_{h \in H^\ell}$ and $\{r_i^\ell(v-1)\}_{i \in I^\ell}$, the step d of PS^E in an ℓ-problem can be characterized by the following equations:

(a) If there are no existing-tenant cycles, define

(a.1) $t_h^\ell(d) := \sup\{t \in [0,1] \mid \; r_h^\ell(d-1) - S_h^\ell(H^\ell(d-1), I^\ell(d-1))(t - t_{d-1}^\ell) > 0\}$ for all $h \in H^\ell(d-1)$;

(a.2) $t_i^\ell(d) := \sup\{t \in [0,1] \mid \; r_i^\ell(d-1) - S_i^\ell(H^\ell(d-1), I^\ell(d-1))(t - t_{d-1}^\ell) > 0\}$ for all $i \in I^\ell(d-1)$;

(a.3) $t_d^\ell := \min\{\min_{h \in H^\ell(d-1)} t_h^\ell(d), \; \min_{i \in I^\ell(d-1)} t_i^\ell(d)\}$;

(a.4) $H^\ell(d) := H^\ell(d-1) \backslash \{h \in H^\ell(d-1) \mid t_h^\ell(d) = t_d^\ell\}$;

(a.5) $I^\ell(d) := I^\ell(d-1) \backslash \{i \in I^\ell(d-1) \mid t_i^\ell(d) = t_d^\ell\}$;

(a.6) $r_h^\ell(d) := r_h^\ell(d-1) - S_h^\ell(H^\ell(d-1), I^\ell(d-1))(t_d^\ell - t_{d-1}^\ell)$;

(a.7) $r_i^\ell(d) := r_i^\ell(d-1) - S_i^\ell(H^\ell(d-1), I^\ell(d-1))(t_d^\ell - t_{d-1}^\ell)$.

[16]In PS two agents are homogeneous if they have the same preferences. But in PS^E two existing tenants are homogeneous only if they have the same preferences and the same private endowment. So for simplicity we replicate existing tenants in large problems.

Here t_h^ℓ is time that h is exhausted and t_i^ℓ is the time that i is full. So step d ends with a house being exhausted or an agent being full, depending on which happens earlier.

(b) If there are existing-tenant cycles, denote the set of them by $C^\ell(d)$. For each $c \in C^\ell(d)$, denote the set of existing tenants and the set of houses involved in c by $c(I)$ and $c(H)$ respectively. Then define

(b.1) $TC(c) = \min\{\min_{h\in c(H)} r_h^\ell(d-1), \min_{i\in c(I)} r_i^\ell(d-1)\}$ for each $c \in C^\ell(d)$;

(b.2) $r_h^\ell(d) = r_h^\ell(d-1) - TC(c)$ if there exists $c \in C^\ell(d)$ such that $h \in c(H)$. Otherwise $r_h^\ell(d) = r_h^\ell(d-1)$;

(b.3) $r_i^\ell(d) = r_i^\ell(d-1) - TC(c)$ if there exists $c \in C^\ell(d)$ such that $i \in c(I)$. Otherwise $r_i^\ell(d) = r_i^\ell(d-1)$;

(b.4) $H^\ell(d) = H^\ell(d-1) \backslash \{h \in H^\ell(d-1) \mid r_h^\ell(d) = 0\}$;

(b.5) $I^\ell(d) = I^\ell(d-1) \backslash \{i \in I^\ell(d-1) \mid r_i^\ell(d) = 0\}$;

(b.6) $t_d^\ell = t_{d-1}^\ell$, and $t_a^\ell(d) = t^\ell(d)$ for $a \in \{h \in H^\ell(d-1) \mid r_h^\ell(d) = 0\} \cup \{i \in I^\ell(d-1) \mid r_i^\ell(d) = 0\}$.

In the following we show that random YRMH-IGYT in the limit problem m^∞ can also be characterized by equations (a.1)-(a.7) and (b.1)-(b.6) when $\ell = +\infty$. This implies that PS^E and random YRMH-IGYT are equivalent in the limit problem.

Specifically, let every agent draw a lottery number uniformly and independently from $[0, 1]$. Then in random YRMH-IGYT agents choose their most preferred houses among remaining ones according to the increasing ordering of their lotteries numbers. We use $\hat{t} \in [0, 1]$ to denote a lottery number, then we say that a step of random YRMH-IGYT ends at time $\hat{t}$ if in expectation a house is exhausted or an agent is full when some agent with a lottery number $\hat{t}$ chooses his most preferred house. By this interpretation we can also track the procedure of random YRMH-IGYT by discrete steps. Then we define $\hat{H}^\infty(d)$, $\hat{I}^\infty(d)$, $\hat{t}_d^\infty$, $\{\hat{r}_h^\infty(d)\}_{h\in H^\infty}$ and $\{\hat{r}_i^\infty(d)\}_{i\in I^\infty}$ for random YRMH-IGYT similarly as we do for PS^E. Then assuming that $H^\infty(d-1) = \hat{H}^\infty(d-1)$, $I^\infty(d-1) = \hat{I}^\infty(d-1)$, $t_{d-1}^\infty = \hat{t}_{d-1}^\infty$, $\{r_h^\infty(d-1)\}_{h\in H^\infty} = \{\hat{r}_h^\infty(d-1)\}_{h\in \hat{H}^\infty}$ and $\{r_i^\infty(d-1)\}_{i\in I^\infty} = \{\hat{r}_i^\infty(d-1)\}_{i\in \hat{I}^\infty}$, that is, step $d-1$ of random YRMH-IGYT coincides with step $d-1$ of PS^E, we want to prove that they also coincide at step d. There are two cases to consider at step d of random YRMH-IGYT:

(1) If there are no existing-tenant cycles, then for any house $h \in \hat{H}^{\infty}(d-1)$, by the proof of Theorem 1 we know that $S_h^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))$ is the mass of agents who either most prefer h, or most prefer the private endowments of some agents who most prefer h, or most prefer the private endowments of some agents who are eventually linked to h. So as long as one of these agents has the chance to consume, h must be consumed. The proportion of these agents who can draw a lottery number between $\hat{t}_{d-1}^{\infty}$ and any $t > \hat{t}_{d-1}^{\infty}$ is exactly $S_h^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))(t - \hat{t}_{d-1}^{\infty})$ by the weak law of large numbers. So the lottery number $\hat{t}_h^{\infty}(d)$ at which h is exhausted is exactly characterized by equation (a.1).

A new agent $i \in \hat{I}_N^{\infty}(d-1)$ can choose his most preferred house in $\hat{H}^{\infty}(d-1)$ only if he draws the smallest lottery number among all agents in $\hat{I}^{\infty}(d-1)$. Since the lottery number is drawn from a uniform distribution, i can obtain a fraction of $t - \hat{t}_{d-1}^{\infty}$ of his most preferred house between $\hat{t}_{d-1}^{\infty}$ and any $t > \hat{t}_{d-1}^{\infty}$. So the lottery number at which i is full is characterized by equation (a.2) where $S_i^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) = 1$. But an existing tenant $j \in \hat{I}_E^{\infty}(d-1)$ can choose his most preferred house either if he draws the smallest lottery number among $\hat{I}^{\infty}(d-1)$, or if he is involved in trading chains which are triggered by a total mass of $S_{\pi^{\infty}(j)}^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))$ other agents who draw the smallest lottery number. By the weak law of large numbers, the chance that i can choose his most preferred house between $\hat{t}_{d-1}^{\infty}$ and any $t > \hat{t}_{d-1}^{\infty}$ is $\left[S_{\pi^{\infty}(j)}^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) + 1\right](t - \hat{t}_{d-1}^{\infty})$. Since $S_j^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) = S_{\pi^{\infty}(j)}^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) + 1$, j can obtain a fraction of $S_j^{\infty}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))(t - \hat{t}_{d-1}^{\infty})$ of his most preferred house. So the lottery number $\hat{t}_j^{\infty}(d)$ at which j is full is also characterized by equation (a.2). Then we can use the remaining equations (a.3)-(a.7) to characterize random YRMH-IGYT.

(2) If there are no existing-tenant cycles, then all the cycles must also appear at step d of PS^E. We trade these cycles immediately in random YRMH-IGYT. So equations (b.1)-(b.6) are also applicable to random YRMH-IGYT.

Since the induction assumption obviously holds at the beginning of the two mechanisms, by induction we prove that PS^E is equivalent to random YRMH-IGYT in the limit problem m^{∞}. Formally, if we denote the assignments found by the two mechanisms by

$PS^E(m^\infty)$ and $\text{RYI}(m^\infty)$, then $||PS^E(m^\infty) - \text{RYI}(m^\infty)|| = 0$.[17] In the on-line appendix we prove that $||PS^E(m^\ell) - PS^E(m^\infty)|| \to 0$ and $||\text{RYI}(m^\ell) - \text{RYI}(m^\infty)|| \to 0$ as $\ell \to +\infty$. So $||\text{RYI}(m^\ell) - PS^E(m^\ell)|| \to 0$ as $\ell \to +\infty$.

B Proofs of Propositions 1-5

Proof of Proposition 1

For any simultaneous eating algorithm satisfying the two conditions (1) and (2), since every two new agents i, i' always have the same set of endowments, it is obvious that $s_i(t) = s_{i'}(t)$ for all $t \in [0, 1]$. Then there are two cases:

• If there are no cycles among existing tenants, then we can normalize the eating speed of every new agent to one. That is, $\sum_{h \in E_i(t)} (s_h(t)/|O_h(t)|) = 1$ for all $i \in I_N$ and all $t \in [0, 1]$. Then for every existing tenant $j \in I_E$, $s_j(t) = \sum_{h \in E_j(t)} (s_h(t)/|O_h(t)|) = s_{\pi(j)}(t) + \sum_{h \in E_i(t)} (s_h(t)/|O_h(t)|) = s_{\pi(j)}(t) + 1$ where i is any new agent.

• If there are cycles among existing tenants, then without loss of generality let a typical cycle be $\pi(j_1) \to j_1 \to \pi(j_2) \to j_2 \to \cdots \to \pi(j_n) \to j_n \to \pi(j_1)$, where every $j_o (o = 1, \ldots, n)$ is an existing tenant. Then "you request my house - I get your speed" implies that $s_{j_1}(t) \le s_{j_2}(t) \le \cdots \le s_{j_n}(t) \le s_{j_1}(t)$. So it must be that $s_{j_1}(t) = s_{j_2}(t) = \cdots = s_{j_n}(t)$. However, we know that $s_{j_2}(t) = s_{\pi(j_2)}(t) + \sum_{h \in E_i(t)} (s_h(t)/|O_h(t)|) \ge s_{j_1}(t) + s_i(t)$ where i is any new agent. So it must be that $s_i(t) = 0$ for any new agent i. Hence for any existing tenant j who is not involved in any cycle, it must be that $s_{\pi(j)} = 0$. So $s_j(t) = s_{\pi(j)}(t) + s_i(t) = 0$. To summarize, only existing tenants in cycles can have positive eating speeds. This is equivalent to trading the cycles immediately.

So a simultaneous eating algorithm satisfying the two condtions (1) and (2) is equivalent to PS^E.

Proof of Proposition 2

Since PS^E can be seen as a simultaneous eating algorithm, it is ordinal efficient. It is obviously individually rational for new agents. It is individually rational for existing tenants because private endowments are exhausted never earlier than their owners stop

[17] $||PS^E(m^\infty) - \text{RYI}(m^\infty)|| = \sup_{i \in I^\infty, h \in H^\infty} |PS^E(m^\infty)_{ih} - \text{RYI}(m^\infty)_{ih}|$ where $PS^E(m^\infty)_{ih}$ and $\text{RYI}(m^\infty)_{ih}$ are the probabilities that i obtains h in $PS^E(m^\infty)$ and $\text{RYI}(m^\infty)$ respectively.

consuming. Lastly, since all new agents always have the same eating speed, there is no envy among them.

Proof of Proposition 3

For any $h \in H$ and any $i \in I$, suppose i reports a strict preference relation $\succsim_i' \in \mathcal{R}$ such that $U(\succsim_i', h) = U(\succsim_i, h)$ and $\succsim_i' |_{U(\succsim_i', h)} = \succsim_i |_{U(\succsim_i, h)}$. Then it is easy to see that the difference between the procedure of PS^E when i reports $\succsim_i$ and the procedure of PS^E when i reports $\succsim_i'$ can happen only after all houses in $U(\succsim_i, h)$ have been exhausted. This implies that the assignments of all houses in $U(\succsim_i, h)$ do not change. That is, $\varphi_{jh'}(\{\succsim_I\}) = \varphi_{jh'}(\{\succsim_i', \succsim_{-i}\})$ for all $j \in I$ and all $h' \in U(\succsim_i, h)$. So PS^E is boundedly invariant.

Suppose $\succsim_i'$ is a dropping strategy of $\succsim_i$ and $U(\succsim_i, \pi(i)) \backslash U(\succsim_i', \pi(i)) = \{h_1, h_2, \ldots, h_\ell\}$. If i does not obtain any fraction of the houses in $\{h_1, h_2, \ldots, h_\ell\}$ by reporting $\succsim_i$, then the outcome of PS^E must keep same no matter i reports $\succsim_i$ or $\succsim_i'$. So without loss of generality we assume h_k is the best house in $\{h_1, h_2, \ldots, h_\ell\}$ that i obtains a positive fraction by reporting $\succsim_i$ but drops in $\succsim_i'$. Since PS^E is boundedly invariant, the fraction of any house better than h_k that i obtains does not change if he reports $\succsim_i'$. However, by dropping h_k agent i must lose the positive fraction of h_k. So the lottery i obtains by reporting $\succsim_i'$ cannot first-order stochastically dominate the lottery i obtains by reporting $\succsim_i$. So PS^E is weakly dropping-strategy-proof.

If $\succsim_i'$ is a truncation strategy of $\succsim_i$, then there exists some $h \succsim_i \pi(i)$ such that $\succsim_i' |_{U(\succsim_i', \pi(i))} = \succsim_i |_{U(\succsim_i, h)}$. As before, the fraction of every house better than h that i obtains does not change if he reports $\succsim_i'$. But by reporting $\succsim_i'$ he fills all of his remaining demand by $\pi(i)$, while by reporting $\succsim_i$ he may obtain a positive fraction of some house strictly better than $\pi(i)$. So the lottery obtained by reporting $\succsim_i$ first-order stochastically dominates the lottery obtained by reporting $\succsim_i'$. Hence PS^E is truncation-strategy-proof.

Proof of Proposition 4

Individual rationality, *new-agent envy-freeness* and *bounded invariance* hold in the same way as before. In the following we prove the remaining properties.

First, we prove *ordinal efficiency*. Suppose the random assignment PS^E under weak preferences finds for some problem is not ordinally efficient. Then there must exist $k \geq 2$ agents $i_1, i_2, \ldots, i_k$, and k houses in their consumption profiles $h_1^{i_1}, h_2^{i_2}, \ldots, h_k^{i_k}$ such that if they trade their consumptions as in the following cycle, none of them is worse off while someone is strictly better off:

$$i_1 \to h_2^{i_2} \to i_2 \to h_3^{i_3} \to i_3 \to \cdots \to h_k^{i_k} \to i_k \to h_1^{i_1} \to i_1,$$

Here $h_o^{i_o}(o = 1, \ldots, k)$ is the consumption of i_o and by trading the cycle, i_o obtains $h_{o+1}^{i_{o+1}}$. Without loss of generality we assume that i_1 is strictly better off while others' welfare does not change. That is, i_1 strictly prefers h_2 to h_1, while $i_o(o = 2, \ldots, k)$ is indifferent between h_o and h_{o+1}. Then in the procedure of PS^E under weak preferences, when i_1 begins consuming h_1 at some step d, h_1 either is not exhausted or has been exhausted but some agent labels his consumption of h_1 as available, and at the same time h_2 must have been exhausted and no agent labels his consumption of h_2 as available. Then since i_k is indifferent between h_1 and h_k, at step d either h_k is not exhausted or it has been exhausted but i_k labels his consumption of h_k as available. By the same argument, since i_{k-1} is indifferent between h_k and h_{k-1}, at step d either h_{k-1} is not exhausted or it has been exhausted but i_{k-1} labels his consumption of h_{k-1} as available. Repeating this argument we know that at step d either h_2 is not exhausted or it has been exhausted but i_2 label his consumption of h_2 as available. However, this contradicts our earlier conclusion. So PS^E under weak preferences is ordinally efficient.

Second, we prove weak dropping-strategy-proofness. Suppose $\succsim_i'$ is a dropping strategy of $\succsim_i$ and $U(\succsim_i, \pi(i)) \backslash U(\succsim_i', \pi(i)) = \{h_1, h_2, \ldots, h_\ell\}$. Let h_k be one of the best houses in $\{h_1, h_2, \ldots, h_\ell\}$ that i obtains a positive fraction by reporting $\succsim_i$ but drops in $\succsim_i'$. Let $H(h_k)$ and $H'(h_k)$ be the set of houses indifferent with h_k in $\succsim_i$ and $\succsim_i'$ respectively. If $H'(h_k) = \emptyset$, then it is obvious that the lottery i obtains by reporting $\succsim_i'$ cannot first-order stochastically dominate the lottery i obtains by reporting $\succsim_i$. If $H'(h_k) \neq \emptyset$, then it must be that $H'(h_k) \subseteq H(h_k)$. Since PS^E under weak preferences is boundedly invariant, i obtains the same fraction of any house better than h_k by reporting either $\succsim_i$ or $\succsim_i'$. But $H'(h_k) \subseteq H(h_k)$ implies that the total fraction of the houses in $H'(h_k)$ that i obtains by reporting $\succsim_i'$ cannot exceed the total fraction of the houses in $H(h_k)$ that i obtains by reporting $\succsim_i$. Then we can repeat the above argument for the remaining houses in $\{h_1, h_2, \ldots, h_\ell\}$ and finally prove weak dropping-strategy-proofness. Truncation-strategy-proofness can be proved in almost the same way.

C PS^{IR}, TTC^E

C.1 The PS^{IR} algorithm.

We show the procedure of PS^{IR} in Example 1.

Step 1: At $t = 0$, every agent eats his most preferred house with speed one. Since the set of acceptable houses for i_1, i_2 is $\{h_1, h_2, h_3\}$, when more than one unit in $\{h_1, h_2, h_3\}$ are eaten by agents other than i_1, i_2, the individual rationality of i_1, i_2 will be violated. So i_3, i_4, i_5, i_6 are blocked from eating h_1, h_2, h_3 after $t = 1/4$. So at $t = 1/4$ the problem is broken into two sub-problems: $m_1 = \{\{1/2h_1, 1/2h_2, 1/2h_3\}, \{i_1, i_2\}\}$ and $m_2 = \{\{h_4, h_5, h_6\}, \{i_3, i_4, i_5, i_6\}\}$.

Step 2: For sub-problem m_1, at $t = 1/4$, i_1, i_2 eat h_2, h_3 respectively with speed one. Then at $t = 1/2$, i_1 obtains another $1/4h_2$ and i_2 obtains another $1/4h_3$. So i_2's residual demand is $1/2$ while his acceptable remaining houses are $\{1/4h_2, 1/4h_3\}$. Then if i_1 continues eating h_2 after $t = 1/2$, i_2's individual rationality will be violated. So at $t = 1/2$ we further break m_1 into two sub-problems $m_{11} = \{\{1/4h_2, 1/4h_3\}, \{i_2\}\}$ and $m_{12} = \{\{1/2h_1\}, \{i_1\}\}$.

Step 3: For each of m_{11}, m_{12}, m_2, the procedure of PS^{IR} coincides with PS.

The procedure of PS^{IR} is summarized as:

time	i_1	i_2	i_3	i_4	i_5	i_6
$1/4$	$1/4h_2$	$1/4h_3$	$1/4h_1$	$1/4h_2$	$1/4h_1$	$1/4h_3$
$+1/4$	$1/4h_2$	$1/4h_3$	$1/4h_5$	$1/4h_6$	$1/4h_6$	$1/4h_4$
$+1/4$	$1/4h_1$	$1/4h_3$	$1/4h_5$	$1/4h_6$	$1/4h_6$	$1/4h_4$
$+1/4$	$1/4h_1$	$1/4h_2$	$1/4h_5$	$1/4h_5$	$1/4h_4$	$1/4h_4$

It is easy to see that PS^{IR} is not boundedly invariant since it forecasts the future procedure. In this example although at $t = 1/4$ agents i_1, i_2 have not revealed their preferences over houses other than h_2, h_3 in the procedure, PS^{IR} predicts that the set of acceptable houses for them is $\{h_1, h_2, h_3\}$. So PS^{IR} blocks i_3, i_4, i_5, i_6 from eating h_1, h_2, h_3 after $t = 1/4$.

C.2 The TTC^E algorithm

Now we show the procedure of applying TTC^E to solve Example 1. It is easy to check that the assignment TTC^E finds coincides with the one PS^E finds.

Initialization: Uniformly assign the shares of h_6 to all agents. So each agent's endowment includes $1/6h_6$ and his private endowment if any.

Step 1: The pointing stage is shown in Figure 1. There is only one existing-tenant cycle: $i_{1,h_1} \to i_{2,h_2} \to i_{3,h_3} \to i_{1,h_1}$. After trading the cycle i_1 gets h_2, i_2 gets h_3 and i_3 gets h_1. They are full and leave the algorithm. Their remaining endowments $3 \times 1/6h_6 = 1/2h_6$ are uniformly assigned to remaining agents. Step 1 ends.

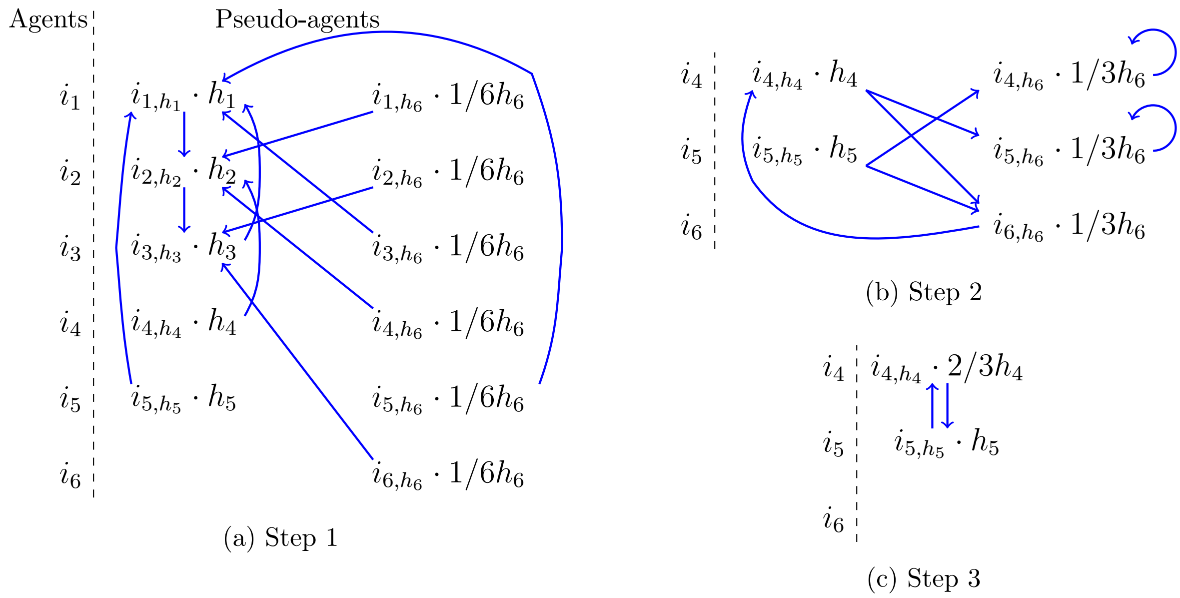


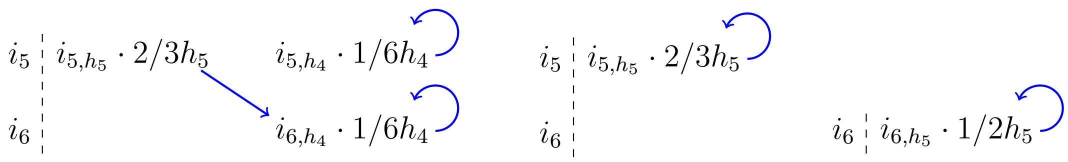
Figure 5: Steps 1, 2 and 3

Step 2: The pointing stage is shown in Figure 1. There are two self-cycles consisting of i_{4,h_6} and i_{5,h_6} pointing to themselves. Since each of them holds $1/3h_6$, the trading quota of both cycles is $1/3$. There is also a feasible new-agent cycle $i_{4,h_4} \to i_{6,h_6} \to i_{4,h_4}$ which involves one new pseudo-agent i_{6,h_6}. The trading quota of this cycle is also $1/3$. After trading these cycles i_4 gets $2/3h_6$, i_5 gets $1/3h_6$, and i_6 gets $1/3h_4$. h_6 is exhausted. Step 2 ends.

Step 3: The pointing stage is shown in Figure 2. There is one existing-tenant cycle $i_{4,h_4} \to i_{5,h_5} \to i_{4,h_4}$. After trading the cycle with quota $1/3$, i_4 gets $1/3h_5$ and i_5

gets $1/3h_4$. Then i_4 is full and leaves the algorithm. His remaining endowment $1/3h_4$ is uniformly assigned to i_5 and i_6. Step 3 ends.

Step 4: The pointing stage is shown in Figure 2. There are two self-cycles consisting of i_{5,h_4} and i_{6,h_4} pointing to themselves. Trade these cycles immediately with quota $1/6$. Step 4 ends.



(a) Step 4 (b) Step 5 (c) Step 6

Figure 6: Steps 4, 5 and 6

Step 5: The pointing stage is shown in Figure 3. There is one self-cycle consisting of i_{5,h_5} pointing to himself. After trading the cycle with quota $1/6$, i_5 obtains $1/6h_5$ and leaves the algorithm. His remaining endowment $1/2h_5$ is assigned to i_6. Step 5 ends.

Step 6: The pointing stage is shown in Figure 3. There is one self-cycle consisting of i_{6,h_5} pointing to himself. After trading the cycle with quota $1/2$, i_6 gets $1/2h_5$ and leaves the algorithm. The algorithm stops.