Level-k Reasoning in School Choice

Jun Zhang*

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Abstract

Many cities use centralized algorithms to assign children to public schools. Recently some cities switch from a manipulable algorithm called Boston Mechanism to a strategy-proof algorithm called Deferred Acceptance. The effect of the switch is not well understood, especially when evidence shows that parents often have different abilities to manipulate BM. In this paper I use the level-k model to study the strategies used by parents of heterogeneous sophistication levels in BM. I find that the level-k reasoning process in BM is analogous to the procedure of DA. This analogy provides a new understanding of BM. In particular, it implies that the assignment of BM is no less efficient than that of DA. By varying the beliefs of parents in the level-k model I find that a child is guaranteed to benefit from his parent’s sophistication in BM only when his parent’s level is high relative to others and his parent’s belief about others’ levels is relatively accurate. Through simulations I show that my model can generate datasets with patterns similar to empirical observations.

Keywords: School choice, Boston Mechanism, Deferred Acceptance, Level-k model

JEL Classification: C78, D61, D78, I21, I28

*Division of Humanities and Social Sciences, California Institute of Technology. Address: 1200 East California Blvd, MC 228-77, Pasadena, CA 91125. Email: jzhang@hss.caltech.edu. I am grateful to Federico Echenique, Leeat Yariv and Kim Border for guidance and conversation. I thank the audiences at Caltech and Midwest Economic Theory conference (Spring 2016) for comments. All errors are mine.
1 Introduction

In this paper I study a new centralized system that many cities of the world use to assign students to schools in K-12 public education, which is known as school choice. It is different from the traditional system in which students are simply assigned to schools according to their home locations. In the new system students (actually their parents) can express their preferences over schools that are not limited to be in the school zones they live in, and students have chances to attend schools outside of their school zones. It is believed that this new system gives parents more control over their children’s education, and improves diversity in schools. In the system students first submit their preference orderings of schools to an office, then the office runs a computer algorithm to find an assignment of students to schools. From the perspective of economics, a computer algorithm is a mechanism that maps the submitted preferences to an assignment, and school choice is actually a game for students.

In this paper I study two algorithms that are widely used by cities: the Boston Mechanism and the student-proposing deferred acceptance. BM was used by Boston before 2005 and is still widely used by many other cities. DA was proposed by Gale and Shapley (1962) and then adapted by Abdulkadiroğlu and Sönmez (2003) to school choice. Their difference is that reporting true preferences is a weakly dominant strategy for students in DA, while in BM students may want to report non-truthful preferences to obtain better assignments, in which sense BM is said to be manipulable. This difference becomes an important reason for some cites to switch from BM to DA. A lot of research compares the two algorithms and attempts to find which one produces a better assignment. But a major difficulty is that the strategies used by students in BM have not been understood well. There is evidence in both experiments and empirical datasets that students are often boundedly rational and they have different abilities to choose best strategies in BM. Specifically, some students do not realize that school choice is a game, so they do not try to manipulate BM; the other students realize it, but some of them play better strategies than the others. Hence, in this paper I want to explore the strategies used by students in BM when they have different sophistication levels to manipulate it, then compare the assignments found by BM and DA.

To model the heterogeneous sophistication of students I use a nonequilibrium model called level-k in the literature. The level-k model was first proposed by Stahl and Wilson
and Nagel (1995), then developed by Ho, Camerer and Weigelt (1998); Costa-Gomes, Crawford and Broseta (2001); Costa-Gomes and Crawford (2006); Crawford and Iriberri (2007a,b); Arad and Rubinstein (2012), and many others. Many experiments have shown that the model has a good explanatory power. In the model every student has a discrete sophistication level, which is his depth of strategic reasoning. If a student’s level is zero, he is naive and reports true preferences. If a student’s level is k for any k>0, he engages in strategic reasoning and chooses a best strategy (i.e., a best ordering of schools) based on his belief about others’ levels. In the paper I consider two extreme settings of the beliefs of all level-k students to check the robustness of my results, and also examine the effect of different belief settings on the welfare of students. In the first setting a level-k student believes that all others are level-k-1. It is the most common setting used by the literature, so I call the corresponding model the *original level-k*. In this setting the belief of any level-k student does not depend on the true levels of the others. By contrast, in the second setting a level-k student knows the true levels of those whose levels are lower than k and believes that the remaining are level-k-1. In the level-k model a level-k student is impossible to know or believe that any other student’s level is k or higher,\(^1\) so in the second setting a level-k student has the most information he can have. Hence, I call the corresponding model the *informational level-k*.

Since strategic reasoning is not needed in DA, students report true preferences even though they have different sophistication levels. So my main task is to analyze the two level-k models of BM. To study the problem in the most transparent environment I assume complete information. That is, students commonly know each other’s preferences and the priority rankings of schools over students. Although this assumption is not realistic, it removes the effect of probabilistic beliefs and risk attitudes in the incomplete information environment on the strategies of students. So I can focus on the effect of heterogeneous sophistication. In the following I briefly discuss my main results from the two level-k models of BM.

First, in both level-k models when students reason about the strategies of the others in BM, they essentially reason about the most preferred schools reported by the others. This is determined by the feature of BM that if a student reports some schools as most

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\(^1\)If a level-k student believes that any other’s level is k or higher, by reasoning about the others’ strategies he behaves as if having a level higher than k.
preferred, he has the best chance to get admitted by the school in BM. Then I find that the reasoning process of each positive-level student about the most preferred schools reported by the others in both level-k models is analogous to the procedure of DA. Specifically, when a positive-level student thinks about his best strategy, it is as if he runs some kind of DA in his mind to decide which school he should report as most preferred in BM, and he runs more rounds of DA in his mind if his level is higher. I illustrate this observation in Section 3 through an example.

Second, I compare the assignments found by the two algorithms and find that in both level-k models the assignment found by BM is not strictly Pareto dominated by the assignment found by DA for any level distribution of students. Specifically, in both level-k models of BM a student always reports a school weakly better than his assignment in DA as most preferred. So it is impossible that all students obtain worse assignments in BM than in DA. When all students have levels above some thresholds I characterize in the paper, all students report their assignments in DA as most preferred, and BM finds the same assignment as DA does. When the strategies of positive-level students further satisfy a mild condition, the assignment of BM is not Pareto dominated by that of DA.

Last, to address the concern in many cities that sophisticated students can take advantage of truth-telling students and obtain better assignments in BM, I examine the relationship between a student’s sophistication level and his welfare in BM. In the original level-k model a student is not guaranteed to obtain a better assignment if he has a higher level. It is because at a higher level he may overestimate some others’ levels and choose an overcautious strategy. By contrast, in the informational level-k model a student never overestimates the others’ levels. When a student’s level is sufficiently high relative to the others, he knows the true levels of most others and chooses a truly best strategy. Then I show that students of sufficiently high levels must have weakly better assignments in BM than in DA, and they do not like any of the remaining student to become more sophisticated. So students of sufficiently high levels have an advantage in BM in the informational level-k model. However, the contrast between the two models implies that both relatively high levels and relatively accurate beliefs are crucial for such advantage to exist.

To quantify the difference between BM and DA, I simulate their assignments by randomly generating the levels of students. The results show that neither algorithm
clearly dominates the other. Specifically, there are always a percentage of students who prefer the assignments in BM and a percentage of students who prefer the assignments in DA, and both percentages are often significantly above zero. The former percentage is often higher than the latter percentage, so more students prefer BM. There are also more students who obtain extreme assignments in BM than in DA. To examine the effect of sophistication in BM, I look at the assignments of students at each sophistication level and compare it with the counterpart in DA. I find that the assignment of an average $L_0$ students in BM is worse than that of an average student of any positive level, and also worse than the assignment of an average $L_0$ student in DA. By contrast, the assignment of an average student of any positive level is better in BM than in DA. In the original level-k model the welfare of students in BM is single-peaked and the peak is at about the mean level, while in the informational level-k model the welfare of students in BM is monotonic in their levels. My simulation results are similar in some respects to recent empirical estimations of He (2014) and Calsamiglia, Fu and Güell (2015). They use practical datasets to estimate the preferences and strategic types of students, and study the effect of replacing BM with DA. Both studies estimate that replacing BM with DA will hurt more students than helping them, and an average student will lose utility. However, they do not estimate the possible heterogeneous sophistication types among strategic students. I hope my models and simulation results can motivate empirical research to address this possibility in future.

In the rest of the paper I first present the school choice model in Section 2. Then I provide an example to illustrate BM and DA and the two level-k models in Section 3. I formally analyze the original level-k model in Section 4 and the informational level-k model in Section 5. I have some discussions in Section 6. I present simulation results in Section 7. I do some extensions in Section 8 and discuss related literature in Section 9. Section 10 concludes. The appendix includes omitted proofs and additional results.

## 2 School Choice Model

A school choice problem consists of the following elements:

- a finite set of students $I$;
- a finite set of schools $S$;
• a capacity vector \( Q_S = \{q_s\}_{s \in S} \) where \( q_s \) is the number of seats at school \( s \);

• a priority profile of schools \( \Pi_S = \{\pi_s\}_{s \in S} \) where \( \pi_s \) is the strict priority ranking of school \( s \) over students;

• a preference profile of students \( P_I = \{P_i\}_{i \in I} \) where \( P_i \) is the strict preference ordering of student \( i \) over schools.

There are enough seats to admit all students such that \( \sum_{s \in S} q_s = |I| \). This can accommodate two cases. First, the law in many cities requires each student attend a public school, so it is natural to assume enough seats. Second, if students have outside options (private schools or studying at home), I let some school \( s \in S \) denote the outside option. An assignment of students to schools is a function \( \mu : I \rightarrow S \) such that \( |\mu^{-1}(s)| \leq q_s \) for all \( s \in S \). Here \( \mu(i) \) is the assignment of each student \( i \) and \( \mu^{-1}(s) \) is the set of students admitted by each school \( s \). I denote the set of all assignments by \( \mathcal{M} \). A student \( i \) justified envies another student \( j \) in an assignment \( \mu \) if \( \mu(j) P_i \mu(i) \) and \( i \pi_{\mu(j)} j \). That is, \( i \) has a higher priority than \( j \) at school \( \mu(j) \) but \( i \) is assigned to a school worse than \( \mu(j) \).

An assignment \( \mu \) is wasteful if \( |\mu^{-1}(s)| < q_s \) and \( s P_i \mu(i) \) for some \( s \) and some \( i \). That is, \( s \) has empty seats and \( i \) prefers \( s \) to his assignment. An assignment is stable if it does not contain justified envies and is not wasteful.

I use \( R_i \) to denote the weak preference ordering associated with \( P_i \). An assignment \( \mu \) Pareto dominates another assignment \( \mu' \) if \( \mu(i) R_i \mu'(i) \) for all \( i \in I \), and \( \mu(j) P_j \mu'(j) \) for some \( j \in I \). If \( \mu(i) P_i \mu'(i) \) for all \( i \in I \), \( \mu \) strictly Pareto dominates \( \mu' \). An assignment is Pareto efficient if it is not Pareto dominated by any other assignment. I use \( \mathcal{P} \) to denote the set of all strict preference orderings over \( S \), and use \( \mathcal{O} \) to denote the set of all school choice problems. \( \Pi_S \) is regulated and known by the school choice office. So throughout the paper I fix \( I, S, Q_S \) and \( \Pi_S \), and denote a school choice problem simply by \( P_I \). A school choice algorithm is a function \( \psi : \mathcal{O} \rightarrow \mathcal{M} \) such that \( \psi(P_I) \) is the assignment found for \( P_I \). \( \psi \) is Pareto efficient or stable if \( \psi(P_I) \) is Pareto efficient or stable for all \( P_I \). \( \psi \) is strategy-proof if all students weakly prefer to reporting true preferences. Formally, \( \psi(P_I)(i) R_i \psi(\{P'_i, P_{-i}\})(i) \) for all \( i \in I \), all \( P_I \in \mathcal{P}^{|I|} \) and all \( P'_i \in \mathcal{P} \).
3 An Example

Consider a school problem that contains three students Alex, Bob, and Charlie, and three schools X, Y and Z. Each school has only one seat. The preferences of students and priority rankings of schools are shown below.

<table>
<thead>
<tr>
<th>Alex</th>
<th>Bob</th>
<th>Charlie</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>Y</td>
<td>Charlie</td>
<td>Bob</td>
<td>Alex</td>
</tr>
<tr>
<td>Y</td>
<td>Y</td>
<td>X</td>
<td>Alex</td>
<td>Alex</td>
<td>Bob</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
<td>Z</td>
<td>Bob</td>
<td>Charlie</td>
<td>Charlie</td>
</tr>
</tbody>
</table>

3.1 Procedures of BM and DA

In school choice each student is required to submit a list of schools, which is supposed to be his preference ordering over schools, to an office. Then the office runs an algorithm to determine which students are assigned to which schools. BM and DA are two most popular algorithms. Both of them are run in multiple rounds. In the first round, the applications of all students are simultaneously sent to the first schools in their reported lists, then schools tentatively admit applicants according to priority rankings. Then the applications of all rejected students are simultaneously sent to the next schools in their reported lists in the next round, and so on. But they have a crucial difference: in BM once a school has no empty seats in some round, it cannot admit new applicants even though some of them have higher priorities than some students it admitted in earlier rounds; but in DA schools only consider the priority rankings of applicants without considering the timing of receiving their applications. For each school choice problem DA finds the student-optimal stable assignment, which Pareto dominates any other stable assignment and denoted by $\mu^{DA}$.

The Procedures of BM and DA

Round 1: Each student applies to the first school in his reported list. Each school tentatively admits its applicants one by one according to its priority ranking until its all seats are occupied or all applicants are admitted. Remaining applicants, if any, are rejected. If all students are admitted, stop the procedure and finalize all assignments.
Round $r \geq 2$: Each rejected student applies to the next school in his reported list.

- In **BM**, each school with empty seats tentatively admits its applicants one by one according to its priority ranking until its all seats are occupied or all applicants are admitted. Remaining applicants, if any, are rejected. If all students are admitted, stop the procedure and finalize all assignments.

- In **DA**, each school that receives new applications considers its earlier admitted students and new applicants and admits them one by one according to its priority ranking until its all seats are occupied or all students are admitted. Remaining students, if any, are rejected. If all students are admitted, stop the procedure and finalize all assignments.

Suppose all students in the above example report true preferences to the school choice office. Then Table 1 shows the procedures of BM and DA by presenting the school each rejected student applies to in each round. In particular, although Bob has a higher priority than Charlie at Y, in BM Bob loses his chance at Y because he applies to Y in the second round while Charlie applies to Y in the first round. However, in DA Bob is admitted by Y since Y ignores the timing of receiving applications.

<table>
<thead>
<tr>
<th>Round</th>
<th>BM</th>
<th>DA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alex</td>
<td>Bob</td>
</tr>
<tr>
<td>1</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td>Y</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Z</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Procedures of BM and DA

In BM if a student ranks a school higher in his reported list, his application will be sent to the school in an earlier round, and he has a higher chance to be admitted. So in BM students may want to report non-truthful preference orderings, especially non-truthful first choices, to obtain better assignments. In the above example if Bob reports Y as first choice, he will be admitted by Y in BM. By contrast, in DA reporting true
preferences is a weakly dominant strategy for each student. So no students can obtain better assignments in DA by reporting non-truthful preferences, and DA is said to be strategy-proof.

3.2 Level-k Model of BM

I use two level-k models to analyze the strategies of students in BM when they have different sophistication levels. In both models a level-0 student is naive and reports true preferences. In the original level-k model a level-k student for any $k > 0$ believes the others are level-k-1 and chooses a best strategy. In the informational level-k model a level-k student knows the true levels of those whose levels are lower than k and believes the others are level-k-1, then chooses a best strategy. In the following I use the above example to illustrate the two level-k models of BM by assuming that the preferences and priority rankings are commonly known by all students.

3.2.1 Original Level-k Model of BM in the Example

- Level 0: If a student is level-0, he reports his true preferences in BM. In particular, he reports his most preferred school as first choice.

- Level 1: If a student is level-1, he believes the others are level-0 and chooses a best strategy. In the complete information environment he knows the true preferences reported by the other students. Then if it is Alex, in his best strategy Alex must report X as first choice since he believes that Bob also reports X as first choice. If it is Bob, in his best strategy Bob must report Y as first choice. Otherwise, he will not be admitted by X but also lose the chance at Y. If it is Charlie, in his best strategy Charlie must report Y as first choice. Otherwise, Charlie will be admitted by another school that he reports as first choice. However, for each student it is uncertain that how he reports the whole preference orderings in his best strategy.

- Level 2: If a student is level-2, he believes the others are level-1 and chooses a best strategy. In the complete information environment by conducting the above level-1 reasoning process in his mind, he knows the first choices reported by the others at level 1. Although he is uncertain about their whole reported preference orderings, it is sufficient for him to choose his best strategy in BM. If it is Alex, Alex knows
that X is the school he wants to obtain by using a best strategy, and he can obtain X for sure by reporting it as first choice. So I assume that Alex just reports X as first choice. If it is Bob, in his best strategy Bob must report Y as first choice since he believes that Charlie also reports Y as first choice. If it is Charlie, in his best strategy he must report X as first choice. Otherwise, he will not be admitted by Y but also lose the chance at X. However, for each student it is still uncertain that how he reports the whole preference orderings in his best strategy.

• Level 3: If a student is level-3, he believes the others are level-2 and chooses a best strategy. In the complete information environment by conducting the above reasoning process in his mind, he knows the first choices reported by the others at level 2. If it is Alex, Alex knows that Z is the school he wants to obtain by using a best strategy, and he can obtain Z for sure by reporting it as first choice. So by my assumption Alex just reports Z as first choice. If it is Bob, Bob knows that Y is the school he wants to obtain by using a best strategy, and he can obtain Y for sure by reporting it as first choice. So by my assumption Bob just reports Y as first choice. If it is Charlie, in this best strategy Charlie must report X as first choice since he believes that Alex also reports X as first choice.

• Level $k \geq 4$: If a student is level-4, he believes the others are level-3 and chooses a best strategy. In the complete information environment by conducting the above reasoning process in his mind, he knows the first choices reported by the others at level 3. By my assumption Alex still reports Z as first choice, Bob still reports Y as first choice, and Charlie still reports X as first choice. It is easy to see that same conclusions also apply to all levels higher than 4.

There are two observations from the example. First, in the level-k reasoning process each student essentially reasons about the first choices reported by the others at each level. This is determined by the feature of BM that students are attracted to manipulate first choices. Second, by looking at the first choices reported by students, the level-k

\[2\text{If Alex believes that Charlie reports the preference ordering of } Y \succ Z \succ X \text{ at level 1, he believes he can also obtain X by reporting the preference ordering of } Y \succ X \succ Z. \text{ Hence, it is not 100% sure that Alex must report X as first choice in his best strategy, and so I make the assumption. Arguments to support the assumption are in Section 6.}\]
reasoning process is analogous to the procedure of DA. Specifically, in Table 2 I put the level-k reasoning process and the procedure of DA side by side. For the original level-k model I list the first choices reported by students at each level. For each student at each round of DA, if the student is admitted by some school in the previous round, I list that school; otherwise I list the school the student applies to in that round. It is easy to see that level 0 of the original level-k model of BM coincides with round 1 of DA, level 1 coincides with round 2, level 2 coincides with round 3, and level 3 and above coincides with round 5. In the paper I formalize this analogy.

<table>
<thead>
<tr>
<th></th>
<th>Alex</th>
<th>Bob</th>
<th>Charlie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0:</td>
<td>X</td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>Level 1:</td>
<td>X</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Level 2:</td>
<td>X</td>
<td>Y</td>
<td>X</td>
</tr>
<tr>
<td>Level 3:</td>
<td>Z</td>
<td>Y</td>
<td>X</td>
</tr>
<tr>
<td>Level $k \geq 4$:</td>
<td>Z</td>
<td>Y</td>
<td>X</td>
</tr>
</tbody>
</table>

(a) Original level-k model of BM

<table>
<thead>
<tr>
<th></th>
<th>Alex</th>
<th>Bob</th>
<th>Charlie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round 1:</td>
<td>X</td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>Round 2:</td>
<td>X</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Round 3:</td>
<td>X</td>
<td>Y</td>
<td>X</td>
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<tr>
<td>Round 4:</td>
<td>Y</td>
<td>Y</td>
<td>X</td>
</tr>
<tr>
<td>Round 5:</td>
<td>Z</td>
<td>Y</td>
<td>X</td>
</tr>
</tbody>
</table>

(b) The procedure of DA

Table 2

3.2.2 Informational Level-k Model of BM in the Example

In the informational level-k model the strategy of a positive-level student depends on the levels of those whose levels are lower than him. Hence, I should assume a specific level distribution to illustrate the model. Suppose Alex is level-1, Bob is level-0, and Charlie is level-2. As before, in the level-k reasoning process students essentially reason about the first choices reported by the others. So in Table 3 I only list the reported first choices at each level.

<table>
<thead>
<tr>
<th></th>
<th>Alex</th>
<th>Bob</th>
<th>Charlie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0:</td>
<td>X</td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>Level 1:</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level 2:</td>
<td>X</td>
<td></td>
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<td></td>
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</tbody>
</table>

Table 3: Informational level-k model of BM
Since Charlie knows that Bob is level-0, he reports Y as first choice and obtains Y in BM. By contrast, in the previous level-k model Charlie would believe that Bob is level-1, so he would report X as first choice and obtain it. So Charlie obtains a better assignment by having a correct belief. In the paper I show that the level-k reasoning process in this model is also analogous to the procedure of DA. I also discuss when the benefit of sophistication exists in BM.

4 The Original Level-k Model of BM

From now on if a student is level-k, I simply say he is $L_k$. In the original level-k model $L_0$ students report true preferences. An $L_k$ student for any $k > 0$ believes the others are $L_{k-1}$. By assuming complete information he can infer the preference orderings reported by the others at lower levels, then chooses a best preference ordering to report. I call the school he wants to obtain by reporting a best preference ordering the best obtainable school at $L_k$ to him. Any preference ordering that lists the best obtainable school at $L_k$ as first choice is a best preference ordering for him. So an $L_k$ student always has multiple best strategies. But as illustrated by the example in Section 3, the student often has to report his best obtainable school at $L_k$ as first choice in all best strategies, especially when he believes that many others also report his best obtainable school at $L_k$ as first choice. When the first choice in his best strategies is not unique (e.g. Alex at $L_2$ in the previous example), I assume that an $L_k$ student always reports his best obtainable school at $L_k$ as first choice, and this is common knowledge among all positive-level students. In Section 6.2 I discuss some arguments to support this assumption.

**Best strategy selection assumption**: An $L_k$ student for any $k > 0$ reports his best obtainable school at $L_k$ as first choice, and this is commonly known by students at positive levels.

In this paper I focus on the first choices reported by students in the level-k reasoning process, and keep silent on the whole preference orderings they may report. This is not only to make my results robust to any assumption on students’ strategies, but also reflects the true situation in the level-k reasoning process that students often are uncertain about the whole preference orderings reported by the others at lower levels. This is illustrated in Section 3.
In the following I slightly adjust the procedure of DA, and prove that the reasoning process in the original level-k model of BM can be understood through running the adjusted procedure. Formally, I use $s^k_i$ to denote the school each student $i$ reports as first choice at any $Lk$. I call the adjusted procedure of DA Fast DA, because it usually runs faster than DA. Specifically, in any round of Fast DA when an unassigned student needs to apply to a new school, he skips any school $s$ that has admitted $q_s$ students who all have higher priorities than him at $s$. But obviously Fast DA and DA always find the same assignment $\mu^{DA}$.

**Fast Deferred Acceptance**

*Round $r \geq 0$: Each unassigned student $i$ applies to his most preferred school $s$ that he has not applied to and has not admitted $q_s$ students who all have higher priorities than $i$ at $s$. Each school tentatively admits students according to its priority ranking. If all students are admitted after this round, stop the procedure.*

I index the first round of Fast DA by $0$ and denote the last round by $r^{FDA}$. Then in Fast DA I define the following notation for each $i$ and each $k \geq 0$:

$$
a^k_i = \begin{cases} 
\text{the school admitting } i \text{ in round } k-1, & \text{if } i \text{ is admitted in round } k-1, \\
\text{the school } i \text{ applies to in round } k, & \text{if } i \text{ is rejected in round } k-1, \\
\mu^{DA}(i), & k > r^{FDA}.
\end{cases}
$$

That is, $a^k_i$ is the school that admits $i$ in round $k-1$ of Fast DA, or is the school $i$ applies to in round $k$. If $k > r^{FDA}$, $a^k_i$ is the school that finally admits $i$ in Fast DA, which is $\mu^{DA}(i)$. Now I prove that $a^k_i$ is exactly the school each $i$ reports as first choice at any $Lk$ of the original level-k model. So it is as if students run the procedure of Fast DA in their minds to do their strategic reasoning.

**Proposition 1.** For any $P_I$, $s^k_i = a^k_i$ for all $i$ and all $k \geq 0$.

Proposition 1 implies that each $i$ must report a weakly worse school as first choice at a higher level, but the school cannot be worse than $\mu^{DA}(i)$.

**Corollary 1.** For any $P_I$ and any $i$,
(1) $s_i^k R_i s_i^{k+1} R_i \mu^{DA}(i)$ for all $k \geq 0$;

(2) there exists some finite $r_i \geq 0$ such that $s_i^k P_i \mu^{DA}(i)$ for all $k < r_i$, and $s_i^k = \mu^{DA}(i)$ for all $k \geq r_i$.

Here I compress the dependence of $r_i$ on $P_I$. There is an intuitive understanding of Corollary 1. When $i$ has a higher level, he also believes that the others have higher levels and the market in BM is more competitive. So $i$ uses a more cautious strategy by reporting a weakly worse school as first choice. In DA students compete with each other only through priority rankings at schools, so the market in DA is most competitive. Since $i$ obtains $\mu^{DA}(i)$ in the most competitive situation, his first choice is never worse than $\mu^{DA}(i)$ in BM.

4.1 Efficiency Comparison of BM and DA

Now I use $k_i$ to denote any $i$’s level and use $k_I \equiv \{k_i\}_{i \in I}$ to denote a general level distribution. If $k_i \geq r_i$ (the threshold defined in Corollary 1), $i$ reports $\mu^{DA}(i)$ as first choice no matter how high $k_i$ is. In this sense I say $i$ is sufficiently sophisticated. Otherwise $i$ is insufficiently sophisticated. I use $\mu^{BM}_{k_I}$ to denote the outcome of BM for any $P_I$.

Although I do not characterize the whole preference orderings reported by positive-level students, characterizing their reported first choices is sufficient for me to make some statements about the comparison of $\mu^{BM}_{k_I}$ and $\mu^{DA}$. Specifically, in the first round of BM there must be some students who are admitted by their reported first choices. By Corollary 1 each $i$ of such students must be admitted by a school strictly better than $\mu^{DA}(i)$ if $i$ is insufficiently sophisticated, and must be admitted by $\mu^{DA}(i)$ if $i$ is sufficiently sophisticated. So I have the following results.

Proposition 2. For any $P_I$:

1. $\mu^{BM}_{k_I}$ is not strictly Pareto dominated by $\mu^{DA}$ for any $k_I$;

2. If each student is sufficiently sophisticated, $\mu^{BM}_{k_I} = \mu^{DA}$;

3. If each student is insufficiently sophisticated, $\mu^{BM}_{k_I}$ is not Pareto dominated by $\mu^{DA}$.

Hence, $\mu^{DA}$ can Pareto dominate $\mu^{BM}_{k_I}$ only when some students are insufficiently sophisticated while the others are sufficiently sophisticated. In the following I prove that
there must exist some insufficiently sophisticated $i$ who reports a non-truthful preference ordering between $\mu_{k_1}^{BM}(i)$ and $\mu^{DA}(i)$.

**Lemma 1.** For any $P_I$ and any $k_1$, if $\mu^{DA}$ Pareto dominates $\mu_{k_1}^{BM}$, there exists some insufficiently sophisticated $i$ who reports some $P'_i$ such that $\mu_{k_1}^{BM}(i) P'_i \mu^{DA}(i)$ but $\mu^{DA}(i) P_i \mu_{k_1}^{BM}(i)$.

So if each insufficiently sophisticated $i$ reports the truthful preference ordering between $\mu^{DA}(i)$ and any school worse than $\mu^{DA}(i)$, then $\mu_{k_1}^{BM}$ is never Pareto dominated by $\mu^{DA}$.

**Proposition 3.** For any $P_I$ and any $k_1$, if each insufficiently sophisticated $i$ reports any $P'_i$ such that $\mu^{DA}(i) P_i s$ implies $\mu^{DA}(i) P'_i s$ for any $s \in S$, then $\mu_{k_1}^{BM}$ is not Pareto dominated by $\mu^{DA}$.

There is a simple strategy $P'_i$ that satisfies the above assumption: if $i$ reports $s$ as first choice, then $s P'_i s'$ for all $s' \neq s$, and $s P'_i s''$ if and only if $s' P_i s''$ for all $s', s'' \neq s$. That is, $i$ only manipulates the first choice and reports the true preference ordering of the remaining schools. So I call $P'_i$ a *topping strategy*.

Given any $P_I$, I call the ordering of the schools that any $i$ applies to in the procedure of Fast DA the *expressed preferences of $i$ in Fast DA*. If $\mu^{DA}$ is not Pareto efficient with respect to the expressed preferences of students in Fast DA, I can construct a level distribution $k_I$ in which all students obtain their reported first choices such that $\mu_{k_1}^{BM}$ Pareto dominates $\mu^{DA}$.

**Proposition 4.** For any $P_I$, if $\mu^{DA}$ is not Pareto efficient with respect to the expressed preferences of students in Fast DA, then there exists some $k_1$ such that $\mu_{k_1}^{BM}$ Pareto dominates $\mu^{DA}$.

### 4.2 Benefit of Sophistication in BM

In school choice a concern about BM is that sophisticated students may manipulate BM better than naive students. In this paper the question becomes whether a student obtains a better assignment in BM if he has a higher level. However, in the original level-$k$ model an $L_k$ student for any $k > 0$ believes the others are $L_{k-1}$. So he may overestimate some students’ levels but underestimate some others’. This incorrect belief may help or hurt him, which is uncertain in general. For example, in Section 3 if all students are level-0, Alex is admitted by X and Bob is admitted by Z. If Bob becomes $L_1$, Bob is admitted
by Y and becomes better off. But if Alex becomes level-3, Alex is admitted by Z and becomes worse off.

For any $P_I$, any sufficiently sophisticated $i$ must be admitted by $\mu^{DA}(i)$ in BM irrespective of the others’ levels.\textsuperscript{3} If all students’ levels are randomly generated, the above can be seen as an advantage of $i$ from an interim view since there is no uncertainty in his assignment. But since $i$ is also admitted by $\mu^{DA}(i)$ in DA, $i$ does not have any additional advantage in BM.

5 The Informational Level-k Model of BM

In the informational level-k model an $L_k$ student for any $k > 0$ knows any $L_{k'}$ student’s level if $k' < k$, and believes the remaining are $L_k - 1$. So his strategy in BM depends on the level distribution $k_I$. This is different from the previous model. Hence, in this section I use $s^k_i(k_I)$ to denote the first choice reported by any $i$ at $L_k$ for any $0 \leq k \leq k_i$. I show that the level-k reasoning process can still be understood through an adjusted procedure of DA. Formally, for any $P_I$ and any $k_I$, I define:

\textbf{Fast Deferred Acceptance*}

\textit{Round $r \geq 0$:} For each unassigned student $i$, if $k_i \geq r$, then $i$ applies to her most preferred school $s$ that he has not applied to and has not admitted $q_s$ students who all have higher priorities than $i$ at $s$. Each school tentatively admits students according to its priority ranking. If $k_i < r$ for all unassigned $i$, or all students are admitted after this round, stop the procedure.

Fast $DA^*$ is different from Fast DA in that an unassigned $i$ cannot apply to a new school in any round $r > k_i$. Since its procedure depends on $k_I$, I denote its outcome by $\mu^{FDA^*}_{k_i}$. If some $i$ is unassigned in $\mu^{FDA^*}_{k_i}$, I say $i$ is admitted by $\emptyset$. Let $r^{FDA^*}_{k_i}$ denote the last round of Fast $DA^*$. Then in Fast $DA^*$ I define the following notation for each $i$ and each $0 \leq k \leq k_i$:

\textsuperscript{3}The proof is very simple. If some students other than $i$ also report $\mu^{DA}(i)$ as first choice in BM, then these students must apply to $\mu^{DA}(i)$ in some rounds of DA. Since $i$ is admitted by $\mu^{DA}(i)$ in DA, $i$ must be admitted by $\mu^{DA}(i)$ in the first round of BM.
\[
\tilde{a}_i^k(k_I) = \begin{cases} 
\text{the school admitting } i \text{ in round } k-1, & \text{if } i \text{ is admitted in round } k-1, \\
\text{the school } i \text{ applies to in round } k, & \text{if } i \text{ is rejected in round } k-1, \\
\tilde{a}_i^{r_FDA^*} & \text{if } k > r_FDA^* \end{cases}
\]

**Proposition 5.** For any \( P_I \) and any \( k_I \), \( s_i^k(k_I) = \tilde{a}_i^k(k_I) \) for all \( i \) and all \( 0 \leq k \leq k_i \).

\( \tilde{a}_i^k(k_I) \) is the last school each \( i \) applies to in Fast DA*, and also the first choice reported by each \( i \) in BM. So \( \mu_{k_I}^{FDA^*} \) is the assignment found by the first round of BM.

**Corollary 2.** For any \( P_I \) and any \( k_I \), \( \mu_{k_I}^{FDA^*} \) is the assignment found by the first round of BM.

If all students are sufficiently sophisticated, each \( i \) must finally apply to \( \mu^{DA}(i) \) in Fast DA*. So \( \mu_{k_I}^{FDA^*} \) coincides with \( \mu^{DA} \). If some \( i \) is insufficiently sophisticated, since he applies to fewer schools than being sufficiently sophisticated, some other \( j \) whose level is higher than \( i \) may therefore only apply to schools better than \( \mu^{DA}(j) \) in Fast DA*.

**Corollary 3.** For any \( P_I \),

1. For any \( k_I \), \( s_i^k(k_I) R_i s_i^{k+1}(k_I) R_i \mu^{DA}(i) \) for all \( i \) and all \( 0 \leq k \leq k_i \);
2. If each student is sufficiently sophisticated, \( \mu_{k_I}^{FDA^*} = \mu^{DA} \);
3. If each student is insufficiently sophisticated, \( s_i^k(k_I) P_i \mu^{DA}(i) \) for all \( i \).

### 5.1 Efficiency Comparison of BM and DA

For any \( P_I \) and any \( k_I \), I denote the outcome of BM by \( \tilde{\mu}_{k_I}^{BM} \). Using Corollary 3 I can prove the following result in the same way as in the previous section.

**Proposition 6.** For any \( P_I \):

1. \( \tilde{\mu}_{k_I}^{BM} \) is not strictly Pareto dominated by \( \mu^{DA} \) for any \( k_I \);

---

\(^4\)In particular, even though \( j \) is sufficiently sophisticated, if his level is not high enough, \( i \) can be unassigned in in Fast DA*. This is different from the previous model in which a sufficiently sophisticated \( j \) must be assigned to \( \mu^{DA}(j) \) in the first round of BM.
2. If each student is sufficiently sophisticated, $\tilde{\mu}_{k_i}^{BM} = \mu^{DA}$;

3. If each student is insufficiently sophisticated, $\tilde{\mu}_{k_i}^{BM}$ is not Pareto dominated by $\mu^{DA}$;

4. If each positive-level $i$ reports some $P'_i$ such that $\mu^{DA}(i) P_i$ implies $\mu^{DA}(i) P'_i$ for any $s \in S$, then $\tilde{\mu}_{k_i}^{BM}$ is not Pareto dominated by $\mu^{DA}$ for any $k_i$.

There is no a counterpart of Proposition 4 in this section because if all students obtain their reported first choice, $\tilde{\mu}_{k_i}^{BM}$ must coincide with $\mu^{DA}$.

5.2 Benefit of Sophistication in BM

To investigate the benefit of sophistication in BM, I compare the outcome of BM when $j$ is $Lk_j$ with the outcome of BM when $j$ is $Lk'_j$ for any $k'_j > k_j$. Since I only characterize the first choices reported by students, I investigate how the assignment found by the first round of BM changes when $j$’s level is increased from $Lk_j$ to $Lk'_j$ for any $k'_j > k_j$. By Corollary 2 it is equivalent to investigating how the outcome of Fast DA* changes. My first result is as follows.

**Proposition 7.** For any $P_I$ and any $k_I$, if any $j \in I$ becomes $Lk'_j$ for any $k'_j > k_j$, then,

- if $\mu_{k_j}^{FDA*}(j) \neq \emptyset$, $\tilde{\mu}_{k_i}^{BM} = \tilde{\mu}_{(k_j', k_{j-1})}^{BM}$;

- if $\mu_{k_j}^{FDA*}(j) = \emptyset$, for any $i \in I$ such that $\mu_{k_i}^{FDA*}(i) \neq \emptyset$ and $\mu_{(k'_j, k_{j-1})}^{FDA*}(i) \neq \emptyset$:
  - if $k_i \leq k_j + 1$, $\tilde{\mu}_{k_i}^{BM}(i) = \tilde{\mu}_{(k'_j, k_{j-1})}^{BM}(i)$;
  - if $k_i > k_j + 1$, $\tilde{\mu}_{k_i}^{BM}(i) \ R_i \tilde{\mu}_{(k'_j, k_{j-1})}^{BM}(i)$.

The proof is as follows. If $j$ is assigned in $\mu_{k_i}^{FDA*}$, it means that $j$ obtains his reported first choice. Then becoming $Lk'_j$ does not change $j$’s strategy as well as the others’. So the outcome of BM does not change. If $j$ is unassigned in $\mu_{k_i}^{FDA*}$, then by becoming $Lk'_j$, $j$ will apply to more schools in Fast DA* than before. Then for any $i$ such that $\mu_{k_i}^{FDA*}(i) \neq \emptyset$ and $\mu_{(k'_j, k_{j-1})}^{FDA*}(i) \neq \emptyset$, if $k_i \leq k_j + 1$, the level change of $j$ cannot affect the set of schools that $i$ applies to in Fast DA*. So $i$’s assignment does not change. If $k_i > k_j + 1$, since $j$ applies to more schools than before in in Fast DA*, $i$ will also apply to weakly more schools than before. So $i$’s assignment must be weakly worse off.
For any $P_I$, define $\bar{r} \equiv \max_{k_I} F^{DA^*}_{k_I}$. That is, $\bar{r}$ is the largest last round of Fast DA for all possible $k_I$.\(^5\) If any $i$’s level is weakly higher than $\bar{r}$, $i$ must be assigned in the outcome of Fast DA irrespective of the others’ levels. So I say $i$ is quasi-rational if $k_i \geq \bar{r}$. A quasi-rational student is sophisticated enough in the sense that he always obtains his reported first choice. For any $P_I$ and any $k_I$, I denote the set of quasi-rational students by $M$ and the set of the remaining by $N$. Proposition 7 implies the following corollary.

**Corollary 4.** For any $P_I$ and any $k_I$, if $M \neq \emptyset$ and $N \neq \emptyset$, then if any $j \in N$ becomes $Lk'_j$ for any $k'_j > k_j$, 

\[
\tilde{\mu}_{k_I}^{BM}(i) R_i \tilde{\mu}_{(k'_j,k_{j-1})}^{BM}(i) \text{ for all } i \in M.
\]

If all students in $N$ become quasi-rational, the outcome of Fast DA will coincide with $\mu^{DA}$. Then Corollary 4 implies the following result.

**Corollary 5.** For any $P_I$ and any $k_I$, if $M \neq \emptyset$, all students in $M$ obtain weakly better assignments in BM than in DA.

For any $k_I$, define $\bar{k}_N \equiv \max_{i \in N} k_i$. If $M = \emptyset$ and there exists a unique $L\bar{k}_N$ student $i$, $i$ must be assigned in $\mu^{F^{DA^*}}_{k_I}$ and obtain an assignment weakly better than $\mu^{DA}(i)$. It is because no students other than $i$ can apply to schools in round $\bar{k}_N$ of Fast DA. Then if $i$ is assigned in round $\bar{k}_N - 1$, $i$ must still be assigned in round $\bar{k}_N$; if $i$ is unassigned in round $\bar{k}_N - 1$, $i$ must apply to a school in round $\bar{k}_N$ and be admitted. So Proposition 7 implies the following corollary.\(^6\)

**Corollary 6.** For any $P_I$ and any $k_I$, if $M = \emptyset$ and there is a unique $L\bar{k}_N$ student $i$,

1. if any $j \in N \setminus \{i\}$ becomes $Lk'_j$ for any $k'_j < k_i$, 

\[
\tilde{\mu}_{k_I}^{BM}(i) R_i \tilde{\mu}_{(k'_j,k_{j-1})}^{BM}(i); 
\]

2. $i$ obtains a weakly better assignment in BM than in DA.

If $\mu^{F^{DA^*}}_{k_I}(j) = \emptyset$, through the following example I show that $j$ is not guaranteed to be better off by becoming more sophisticated. It is because the other students in $N$ who have higher levels than $j$ may respond to the level change of $j$ by using more competitive strategies.

**Example 1.** $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Each school has one seat. The preferences of students and the priority rankings of schools are shown in Table 4.

\(^5\)Given the set $I$ of students and the set $S$ of schools, $\bar{r} \leq |I| \cdot |S| - 1$.

\(^6\)If there are multiple $L\bar{k}_N$ students, the corollary may not hold.
Suppose $i_1$ is $L_0$, $i_2$ is $L_2$, and all others are quasi-rational. The first choices reported by students are shown in Table 5a. If $i_1$ becomes quasi-rational, the first choices reported by students are shown in Table 5b.

<table>
<thead>
<tr>
<th>$P_{i_1}$</th>
<th>$P_{i_2}$</th>
<th>$P_{i_3}$</th>
<th>$P_{i_4}$</th>
<th>$P_{i_5}$</th>
<th>$P_{i_6}$</th>
<th>$\pi_{s_1}$</th>
<th>$\pi_{s_2}$</th>
<th>$\pi_{s_3}$</th>
<th>$\pi_{s_4}$</th>
<th>$\pi_{s_5}$</th>
<th>$\pi_{s_6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
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<td>$s_4$</td>
<td>$s_1$</td>
<td>$s_1$</td>
<td>$i_6$</td>
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<td>$s_2$</td>
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<td>$s_6$</td>
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<td>$i_1$</td>
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</tr>
</tbody>
</table>

Table 4

Level 0: $s_1$ $s_2$ $s_3$ $s_4$ $s_1$ $s_1$

Level 1: $s_2$ $s_3$ $s_4$ $s_4$ $s_1$ $s_1$

Level 2: $s_2$ $s_3$ $s_3$ $s_4$ $s_1$ $s_1$

Level $k \geq 3$: $s_2$ $s_3$ $s_4$ $s_1$ $s_1$

(a) $i_1$ is $L_0$

(b) $i_1$ is quasi-rational

Table 5

If all positive-level students use topping strategies, then the outcomes of BM are shown in Table 6. It is easy to see that $i_1$ is worse off by becoming quasi-rational.

<table>
<thead>
<tr>
<th>$i_1$</th>
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<th>$i_3$</th>
<th>$i_4$</th>
<th>$i_5$</th>
<th>$i_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_5$</td>
<td>$s_6$</td>
<td>$s_2$</td>
<td>$s_3$</td>
<td>$s_4$</td>
<td>$s_1$</td>
</tr>
</tbody>
</table>

(a) $i_1$ is $L_0$

(b) $i_1$ is quasi-rational

Table 6

But if $j$ has the highest level among $N$, $j$ must be weakly better off by becoming more sophisticated and using a strategy that satisfies a mild condition. Formally, I use $P_{kj}^j$ and $P_{kj}'^j$ to denote the preference orderings reported by $j$ at $Lk_j$ and $Lk'_j$ respectively. Then
if \( s \) is the first choice in \( P_j^{k_j} \), I say \( P_j^{k_j} \) satisfies *worse-rank invariance* if for any \( s' \) such that \( sP_jeq s' \), \( s' \) has the same rank in \( P_j^{k_j} \) and \( P_j^{k_j'} \).

**Proposition 8.** For any \( P_I \) and any \( k_I \), if any \( j \in N \) of \( k_j = \bar{k}_N \) becomes \( Lk_j' \) for any \( k_j' > k_j \) and his strategy satisfies worse-rank invariance, then

\[
\tilde{\mu}_{BM}(k_{j'}, k_{-j}) R_j\tilde{\mu}_{BM}(k_I, j).
\]

If \( j \) uses topping strategies at both \( Lk_j \) and \( Lk_j' \), the worse-rank invariance condition is satisfied. If \( j \) becomes quasi-rational, since he must be admitted by his reported first choice, \( j \) is weakly better off even if his strategy does not satisfy worse-rank invariance.

## 6 Discussion

### 6.1 Insights from the Two Level-k Models

In both level-k models of BM the iterated reasoning process is analogous to the procedure of DA. Then I show that in general BM is not (strictly) Pareto dominated by DA. Since the two models make extreme assumptions on the beliefs of positive-level students, I believe similar results will hold in any other level-k model. However, the two models are different in the existence of the advantage of sophisticated students. In the original level-k model sophisticated students do not have definite advantage in BM because they may overestimate the others’ levels. While in the informational level-k model a student has a definite advantage in BM only when his level is high relative to the others. The two model together imply that both high sophistication and an accurate belief are crucial for a student to have an advantage in BM.

### 6.2 Comparison with Nash Equilibrium Models

It is interesting to compare my results with those of Nash equilibrium models. By assuming all students are rational, Ergin and Sönmez (2006) prove that every NE outcome of BM is a stable assignment with respect to true preferences of students. Since DA always finds the student-optimal stable assignment, BM is weakly Pareto dominated by DA. However, in the two level-k models if students are sufficiently sophisticated, BM finds the same assignment as DA. Hence, even though students are very sophisticated, if they
are more likely to use the level-k reasoning than using the circular equilibrium reasoning, they are more likely to coordinate on the stable-optimal stable assignment.

Pathak and Sönmez (2008) use a NE model to show that rational students take advantage of naive students in BM. In their model students are either rational and naive, and rational students commonly know the identifies of naive students. By assuming that the best NE outcome of BM is always realized, they prove that there exists a conflict of interest between naive students and rational students. In particular, rational students obtain weakly better assignments in BM than in DA. This dichotomous sophistication distribution can be seen as a special case of my models. Indeed, I prove that if students are either $L_0$ or quasi-rational, then the outcome of BM in the informational level-k model is exactly the best NE outcome of BM if quasi-rational students are treated as rational. Then the results of Pathak and Sönmez are corollaries of mine in the informational level-k model.

**Proposition 9.** For any $P_I$ and any $k_I$, if $N, M \neq \emptyset$ and $k_N = 0$, then $\tilde{\mu}^{BM}_{k_I}$ is the best NE outcome of BM when $N$ are naive and $M$ are rational.

In Appendix B I use a new method to characterize the set of NE outcomes of BM when $N$ are naive and $M$ are rational. Proposition 9 is a corollary of the characterization. Then by Proposition 7, if any $j \in N$ becomes quasi-rational, all students in $M$ are weakly worse off in BM. Since $k_N = 0$, each $j \in N$ has the highest level in $N$. So by Proposition 8 if any $j \in N$ becomes quasi-rational, $j$ must be weakly better off in BM. Hence I obtain the main results of Pathak and Sönmez.\(^7\)

Abdulkadiroğlu, Che and Yasuda (2011) analyze a special incomplete information environment in which schools do not prioritize students and students share a common preference ordering over schools but may have different cardinal utilities. They prove that if students play any symmetric Bayesian NE of BM, then they have weakly higher utilities in BM than in DA. If there exist naive students, they can benefit from the existence of rational students.\(^8\) The driving force behind their results is the special priority and

\(^7\) In Appendix B I show that the other results of Pathak and Sönmez are also proved easily by my method.

\(^8\) Abdulkadiroğlu, Che and Yasuda (2011) also consider the complete information environment with strict priorities. They prove that if any naive student becomes rational, the other naive students must be weakly worse off in the unique NE outcome of BM. So naive students suffer from the existence of rational students. In Appendix E I show that this result is actually incorrect.
preference assumption, not the incomplete information environment. Specifically, in the no priorities environment any two students of same cardinal utilities are assumed to play same strategies and also treated equally by schools. So a student does not need to know the identities of the others if he knows the distribution of the cardinal utilities in the student population. Hence, assuming common knowledge of the cardinal utility distribution in the incomplete information environment is similar to assuming complete information. In Section 8 I provide a preliminary analysis of the original level-k model of BM in the incomplete information environment.

6.3 Discussion of My Assumptions about Students’ Strategies

$L_0$ strategy I assume that $L_0$ students report true preferences. It is different from the literature that often assume $L_0$ players play a random strategy. The difference is caused by different features of the games. Intuitively, $L_0$ strategy captures the instinct response of a player to a game. Both BM and DA are preference revelation games. So it is very natural for students to report their true preferences if they do not attempt to manipulate the algorithms. However, in some games it is unclear what strategy is a reasonable instinct response. So the literature often assumes that a random strategy is used. For example, in the “p-beauty contest” game it is often assumed that $L_0$ players uniformly choose an integer.\(^9\)

$L_k$ strategy I assume that an $L_k$ student for any $k > 0$ manipulates BM through misreporting first choice. As shown in the example of Section 3, in many situations students have to report their best obtainable schools as first choices if they want to

\(^9\)In the game each player is asked to propose an integer between 0 and 100. The winner is the one whose proposal is closest to a multiple $p$ of the group average.

In some games of the literature it is believed that some strategies are more likely to become instinct responses than the others, and they are called salient strategies. For example, Crawford and Iriberri (2007a) point out the framing effects in the experiments of “hide-and-seek” games. By suitably adapting $L_0$ behavior to salient strategies, they show that the level-k model can well explain the experimental dataset. Arad and Rubinstein (2012) conduct experiments of the “11-20” game to estimate the levels of players. In the game each of two players reports an integer between 11 and 20 and obtains an amount of dollars equaling his report; a player can win additional 20 dollars if his report is one less than the other’s. Since the game rule is straightforward, Arad and Rubinstein argue that it is very natural for a naive player to report 20.
obtain them. In some situations that it is not the case, I believe my assumption is still reasonable. First, because first choices play the most role in determining the assignments of students in BM, students are attracted to optimally report their first choices. It is supported by the experiment of Chen and Sönmez (2006) in which 70.8% of students receive their reported first choices in BM, but only 28.5% receive their true first choices. So over 40% of students manipulate their first choices. Second, in practice students may be advertised to optimally report first choices. For example, Boston provided a reference material to students in 2004 that suggested students to strategically choose their first choices. In Seattle and Tampa-St. Petersburg similar suggestions appear in local press (Abdulkadiroğlu et al., 2005). Last, as shown before, in the level-k reasoning process an $L_k$ student for any $k > 1$ is uncertain about the whole preferences reported by the others at lower levels. If he is risk-averse and considers the worst case, he should assume that the others optimally manipulate their first choices. So my assumption captures the reasoning of an $L_k$ student about the others’ strategies.

7 Simulation

In previous sections I consider all possible level distributions of students. When I define sufficient sophistication and quasi-rationality, the requirements on levels can be high. However, many experiments have found that subjects’ levels are often not high. So in this section I do simulations by taking this fact into account.

7.1 Setup

There are 1000 students and 20 schools. Each school has 50 seats. I define the utility functions of students and schools to generate their preferences and priority rankings. The utility function $U_i$ of each student $i$ and the utility function $U_s$ of each school $s$ are defined as:

$$U_i(s) \equiv \alpha U(s) + (1 - \alpha)\epsilon_i(s),$$
$$U_s(i) \equiv \beta U(i) + (1 - \beta)\epsilon_s(i).$$

Here $U(s)$ and $U(i)$ are the common values of each $s$ and each $i$ respectively. $\epsilon_i(s)$ is $i$’s private value of each $s$ and $\epsilon_s(i)$ is $s$’s private value of each $i$. All $U$ and $\epsilon$ are
independently and identically drawn from the uniform distribution on \([0, 1]\). \(\alpha, \beta \in [0, 1]\)
are correlation coefficients among the utilities of students and the utilities of schools. In
the simulation I vary \(\alpha, \beta\) from 0 to 1 in steps of .2. Hence \(\alpha, \beta \in \{0, .2, .4, .6, .8, 1\}\). The
preferences of students and the priority rankings of schools are generated as:

\[ P_i : s_a P_i s_b \iff U_i(s_a) > U_i(s_b), \]
\[ \pi_s : i_a \pi_s i_b \iff U_s(i_a) > U_s(i_b). \]

For each value of \((\alpha, \beta)\) I randomly generate 1000 markets. In each market I draw the
levels of students independently from the Poisson distribution with a mean of 2. This
distribution is consistent with the estimation in multiple experiments.\(^\text{10}\) In particular,
the probabilities for \(L0\) to \(L4\) are respectively .135, .271, .271, .180, .090. In previous
sections I do not assume how positive-level students report their whole preferences in BM.
In this section I consider two settings to check the robustness of the simulation result.
In the first setting positive-level students use topping strategies. That is, they report
ture preferences over the schools other than reported first choices. In the second setting
they report random preference orderings over the schools other than reported first choices
which are independently drawn from the uniform distribution. I call it random strategy.

7.2 Result

I first use \((\alpha, \beta) = (.4, .4)\) as an example to report the simulation results. The results for
other values of \((\alpha, \beta)\) are similar. To measure the welfare of students in the outcomes of
BM and DA I calculate the ranks of their assignments in their true preferences. Table 7
reports the rank distribution and the average rank in the two level-k models of BM
and DA. It also reports the percentage of students that obtain better assignments in BM
and the percentage of students that obtain better assignments in DA. There are three
observations from the table. First, the rank distributions in the two level-k models of BM
are almost same, and the difference between the topping strategy setting and the random
strategy setting is small. Second, BM produces more extreme assignments than DA.
Specifically, in BM more students obtain very high or low-ranked assignments, while in

\(^\text{10}\) Camerer, Ho and Chong (2004) use the Poisson Cognitive Hierarchy model to estimate multiple
games and find the median estimation of the Poisson mean is 1.61. Arad and Rubinstein (2012) find the
best estimate of the Poisson mean for the “11-20” game is 2.36.
<table>
<thead>
<tr>
<th>Rank</th>
<th>BM (%) (topping)</th>
<th>BM (%) (random)</th>
<th>DA (%)</th>
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<td>25.62</td>
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<th>BM (%) (random)</th>
<th>DA (%)</th>
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<td>20</td>
<td>.38</td>
<td>2.51</td>
<td>&gt; .05</td>
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</tbody>
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Avg rank | 4.0 | 4.9 | 4.6

Topping: 32.8% prefer BM > 12.6% prefer DA
Random: 30.7% prefer BM > 18.2% prefer DA

(a) Original level-k

Topping: 35.1% prefer BM > 14.1% prefer DA
Random: 32.5% prefer BM > 21.1% prefer DA

(b) Informational level-k

Table 7: The rank distribution in BM and DA when \((\alpha, \beta) = (.4, .4)\)

DA more students obtain medium-ranked assignments. Third, although the comparison between BM and DA in terms of the average welfare of students depends on the strategies of positive-level students, there are always more students who prefer BM than those who prefer DA. These observations imply that neither of BM and DA dominates the other, but more students may prefer the assignments in BM.
To examine the effect of sophistication levels on the welfare of students, in Table 8 I report the average rank of the assignments of students at each level from $L_0$ to $L_5$, and in Table 9 I report the percentage of students at each level from $L_0$ to $L_5$ that obtain better assignments in BM or in DA. In DA the average rank is almost always 4.59, which is independent of level. It is not surprising since the strategies of students in DA do not depend on level. However, in BM the average rank obviously depends on level. In particular, the level-0 average rank in BM is lower than any positive-level average rank in BM, and also lower than the level-0 average rank in DA. So level-0 students on average
are worse off in BM than in DA, and they suffer from their naivety in BM. By contrast, the students of any positive level on average are better off in BM than in DA. These can also be observed from Table 9. Among level-0 students there are more who prefer DA than those who prefer BM, while among students of any positive level there are more who prefer BM than those who prefer DA.

However, Table 8 shows an important difference between the two level-k models. In the original level-k model of BM the average rank is not monotonic in level; it has a single peak at $L_3$. As discussed before, students of high levels overestimate the levels of most other students, so they do not choose the true optimal strategies. Under the Poisson distribution with a mean of 2, level-3 students have a correct belief about the levels of most other students, so they enjoy the most benefit from their sophistication. By contrast, in the informational level-k model the average rank is monotonic in level. So higher-level students on average obtain better assignments. Because very few students have levels above $L_3$, the table shows that the marginal benefit from a level above $L_3$ is very small.

I use Figure 1 to summarize the main simulation results for all values of $(\alpha, \beta)$ in the original level-k model. The simulation results for the informational level-k model are almost same and reported in Appendix C.\textsuperscript{11} In each subfigure the horizontal axis is the value of $\alpha$, and the six lines correspond to the six values of $\beta$. Figure 1 shows that for most values of $(\alpha, \beta)$ there are more students who prefer BM than those who prefer DA.\textsuperscript{12} Average rank difference is to compare the average welfare of students in BM and DA. It is equal to the average rank of all students’ assignments in DA minus the average rank of all students’ assignments in BM. If it is positive, students are on average better off in BM.

The figure shows that when topping strategies are used, students are on average better

\textsuperscript{11}Ashlagi, Kanoria and Leshno (2015) show that unbalanced markets perform very differently from balanced markets. In Appendix C I also report the simulation results for unbalanced markets by setting $U(s) = \epsilon_i(s) = 0$ for some $s$ and all $i$. In this way $s$ becomes the worst school in all students’ preferences and plays the role of “unassigned”. Unbalanced markets make some of the simulation results sharper, but my qualitative conclusions do not change.

\textsuperscript{12}$\alpha = 1$ in the topping strategy setting is an exception. It is because when $\alpha = 1$ students have identical preferences, and by using topping strategies they report highly correlated preferences in BM. So the assignment in BM is mainly determined by priority rankings of schools, which is further determined by $\beta$. By contrast, in the random strategy setting students report weakly correlated preferences.
off in BM for all values of $(\alpha, \beta)$\textsuperscript{13}. But when random strategies are used, the answer depends on the values of $(\alpha, \beta)$. So in general it is uncertain that which algorithm gives students a higher average welfare. The last two subfigures are to examine the advantage of sophistication in BM. Average level difference is equal to the average level of those who prefer BM minus the average level of those who prefer DA. In the figure the difference is always positive. Corr. of rank & level reports the correlation coefficient between the preference ranks of students’ assignments in BM and their levels. In the figure it is always positive and significantly above zero for most values of $(\alpha, \beta)$. So both subfigures suggest that students on average benefit from their sophistication in BM.

\textsuperscript{13}When $\alpha = 1$, students have identical preferences, and any assignment has the same average rank. So the average rank difference is zero. While when $\alpha = 0$, students have uncorrelated preferences. Then almost all of them obtain their most preferred schools in both BM and DA. So the average rank difference is almost zero for any $\beta$. 

Figure 1: Simulation results in original level-k
7.3 Comparison with Empirical Estimation

It is interesting to compare the above simulation results with recent empirical estimations conducted by He (2014) and Calsamiglia, Fu and Güell (2015). He uses the dataset from Beijing of China, and Calsamiglia et al. use the dataset from Barcelona of Spain. Both cities implemented some kind of BM in their school choice programs. The two studies accommodate the fact that students have heterogeneous sophistication types. After estimating the preferences of students they conduct counter-factual analyses to predict the effect of replacing BM with DA in the two cities. Specifically, He develops an approach to estimate the preferences of students without having to estimate their sophistication distribution. In his counter-factual analysis he only considers the welfare of naive students and rational students. Calsamiglia et al. estimate both the preferences of students and their sophistication types. But they assume that there are only two sophistication types: being naive or strategic. The results of the two studies are summarized in Table 10. Both studies predict that replacing BM with DA will hurt more students of any sophistication type than benefiting them, and an average student of any sophistication type will have a welfare loss equivalent to either some increase in school distance or some increase in school fee.

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<td></td>
<td>naive</td>
<td>rational</td>
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<tr>
<td>Benefit</td>
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<td>15%</td>
</tr>
<tr>
<td>Hurt</td>
<td>55%</td>
<td>66%</td>
</tr>
<tr>
<td>Average loss</td>
<td>8% ↑ school</td>
<td>40% ↑ school</td>
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<tr>
<td></td>
<td>distance</td>
<td>distance</td>
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Table 10: Empirical estimation of the effect of replacing BM with DA

My simulation results are consistent with the two studies in that we all predict that there are more sophisticated students who will be hurt by the replacement than those who will be helped, and the average welfare of sophisticated students will be reduced by the replacement. However, my simulations predict that the replacement will help naive students in general. The only exception happens when \( \alpha = 0 \), that is, students have uncorrelated preferences. One possible reason for this difference is that the above
studies do not fully consider the possible multiple sophistication types among strategic students. I hope my models and simulations can motivate future empirical research to address this problem.

8 Extension

8.1 Constrained School Choice

There are often many schools in a city, but in some cities students are constrained to report only a few schools in their submitted preferences (Haeringer and Klijn, 2009; Calsmiglia, Haeringer and Klijn, 2010). Under this constraint it is impossible for students to report true preferences in any algorithm. In particular, students cannot report true preferences in DA and may need to do strategic reasoning. So there should be a level-k model of DA. Any such model needs to specify the default strategies of \( L_0 \) students and the best strategies of \( L_k \) students. But without any assumption it is unclear which specification is reasonable. In Appendix D I analyze an original level-k model of DA by assuming that \( L_0 \) students report true preferences truncated by the constraint and \( L_k \) students use topping strategies. I show that the reasoning process in this model is very similar to the one in BM. The analysis of the previous level-k models of BM does not change under the constraint since students essentially reason about each other’s first choice, which is not constrained.

8.2 Incomplete Information

In the previous strict priorities and complete information environment students manipulate BM by misreporting first choices. However, in practice it is often observed that students also misreport other preferences such as the second and the third choices. This is because the practice is an incomplete information environment. So in this section I present the original level-k model of BM in such an environment. To simplify the analysis

\[14\] Although He potentially considers it, he does not model the sophistication distribution of strategic students. In his second counter-factual analysis of replacing BM with DA, He assumes that students are either naive or rational. He finds that the result depends on the proportion of naive students and is different from his first counter-factual analysis. So he concludes that it is important to allow additional sophistication types beyond naivety or rationality.
I assume that schools do not exogenously prioritize students and draw priority rankings randomly from uniform distributions. This is to capture the fact that schools often rank students by a few priority tiers and break the ties within each tier by lotteries.

Denote the cardinal utility vector of each student $i$ by $\mathbf{v}^i \equiv (v^i_s)_{s \in S}$ where $v^i_s$ is the utility of obtaining $s$. $\mathbf{v}^i$ is also called the type of $i$. $\mathbf{v}^i$ is drawn from the type space $\mathcal{V} \equiv \{(v_s)_{s \in S} \in [0, 1]^{|S|} : v_s \neq v_{s'}, \forall s, s' \in S\}$ according to a probability distribution $f$. I assume $f$ is public information and has full support. That is, $f(\mathbf{v}^i) > 0$ for all $\mathbf{v}^i \in \mathcal{V}$. Let $P_v$ be the preference ordering induced by any $\mathbf{v} \in \mathcal{V}$.

I use $P^k_v$ to denote the preferences reported by any type-$\mathbf{v}$ students at any $Lk$. As before $L0$ students report true preferences. So $P^0_v = P_v$. I assume positive-level students are risk-neutral, so they choose strategies to maximize their expected utilities. Then for any $\mathbf{v} \in \mathcal{V}$ and any $k > 0$,

$$P^k_v \equiv \arg\max_{P \in \mathcal{P}} \text{EU}^k_v(P^*),$$

where $\text{EU}^k_v(P^*)$ is the expected utility of type-$\mathbf{v}$ students by reporting $P^*$. Specifically, let $\mu^{BM}(P^{k-1}_v, P^*)$ be the random outcome of BM if a type-$\mathbf{v}$ students $i$ reports $P^*$ and the others report $P^{k-1}_v$. Let $\mu^{BM}(P^{k-1}_v, P^*)(i)(s)$ be the probability that $i$ obtains any $s \in S$. Then,

$$\text{EU}^k_v(P^*) = \int_{\mathbf{v}^{-i} \in \mathcal{V}^{[l]-1}} \sum_{s \in S} [\mu^{BM}(P^{k-1}_v, P^*)(i)(s) \cdot v_s] f(\mathbf{v}^{-i}) d\mathbf{v}^{-i}$$

$$= \sum_{s \in S} \left[ \int_{\mathbf{v}^{-i} \in \mathcal{V}^{[l]-1}} \mu^{BM}(P^{k-1}_v, P^*)(i)(s) f(\mathbf{v}^{-i}) d\mathbf{v}^{-i} \right] v_s$$

When $k = 1$, $P^0_{v^{-i}}$ are the true preferences of the students other than $i$. So $\text{EU}^1_v(P^*)$ is well-defined. Since $f(\mathbf{v}^{-i}) = \prod_{j \neq i} f(\mathbf{v}^j) > 0$ for all $\mathbf{v}^{-i} \in \mathcal{V}^{[l]-1}$, with probability one there is a unique $P^1_v$ that maximizes $\text{EU}^1_v(P^*)$. If $P^1_v$ is not unique, choose an arbitrary best strategy. When $k \geq 2$, since $f(\mathbf{v}^{-i}) > 0$ for all $\mathbf{v}^{-i} \in \mathcal{V}^{[l]-1}$ and $P^{k-1}_v$ is generally unique, $\text{EU}^k_v(P^*)$ is still well-defined. Then $P^k_v$ is still generically unique. If it is not unique, choose an arbitrary best strategy.

As shown above, the strategies of students depend on their beliefs and cardinal utilities. Without any additional assumptions it is hard to characterize their strategies and compare the outcome of BM with that of DA. I leave it for future research.
9 Related Literature

There are a lot of related papers in matching theory and school choice in particular. In Section 6 I have discussed some NE models of school choice. There are still some other papers belonging to this strand. Troyan (2012) generalizes the idea of Abdulkadiroğlu, Che and Yasuda (2011) by relaxing the no priorities assumption to coarse priorities and using some ex ante efficiency criterion. He shows that students are ex ante weakly better off in any symmetric Bayesian NE of BM than in DA. Featherstone and Niederle (2014) use experiments to test the idea of Abdulkadiroğlu et al.. They design a simple environment in which there is a unique non-truth-telling Bayesian NE in BM. But they find that subjects fail to coordinate on the unique equilibrium even with feedback and repetition. It implies that benefit of BM suggested by Abdulkadiroğlu et al. is hard to realize in practice. Instead, they suggest that implementing truth-telling as an ordinal Bayesian NE in BM is an effective way to obtain higher efficiency than DA. Haeringer and Klijn (2009) analyze the NE outcomes of popular matching algorithms in the context of school choice when the reported preferences of students are constrained. They prove that the set of NE outcomes of BM is equal to the set of stable assignments, but the set of NE outcomes of DA is a superset of stable assignments. So it is hard to compare BM and DA by their NE outcomes. Calsamiglia, Haeringer and Klijn (2010) use experiments to study the impact of constraints. They find that constraints significantly reduce the efficiency of BM and DA as well as the proportion of truth telling in DA. But DA is more efficient and stable than BM.

Basteck and Mantovani (2016) and Dur, Hammond and Morrill (2015) use experimental and empirical datasets respectively to examine the advantage of sophisticate students in BM. Specifically, Basteck and Mantovani first measure the cognitive abilities of subjects by standard tests in labs, then let them play the BM and DA games. Although test scores have a wide range, they classify subjects into two groups: high-ability group of top half scores and low-ability group of the remaining. By matching subjects’ performance in the two games with their groups, Basteck and Mantovani find that the low-ability group have significantly lower payoffs than the high-ability group in BM, but the difference is small in DA. Meanwhile, the average payoff of all subjects is higher in BM than in DA. Dur et al. obtain an interesting dataset from Wake County of North Carolina. In the city students have two weeks to submit or revise their preferences as many times as
they want through an online system. Once a student logs into the system, he can see the number of students who has reported each school as first choice. In the dataset an average student visits the system 4.61 times with a standard deviation of 8.65; 60.7% of students visit the system more than once. Dur et al. interpret those who visit once as naive and interpret the remaining as sophisticated. They find that sophisticated students have better assignments than naive students.

Agarwal and Somaini (2014) find that BM is significantly manipulated in the dataset from Cambridge of Massachusetts. So they estimate the preferences of students by assuming all of them are strategic. They predict that replacing BM with DA will make students on average worse off. Top Trading Cycle (Shapley and Scarf, 1974) is a another popular matching algorithm, but is not widely used in school choice. One reason is that it is hard to explain the role of priorities in TTC to schools and students (Pathak, 2016). So I do not analyze it in the paper. Since TTC is strategy-proof, students report true preferences even though they have heterogeneous sophistication levels. Bade (2016) is the only paper I am aware of that discusses the properties of matching algorithms when players are boundedly rational. Specifically, Bade studies the problem of assigning indivisible objects to players in the no priorities environment, and examines whether the large set of hierarchical exchange algorithms are still Pareto efficient.

10 Conclusion

In this paper I study how students behave in a manipulable school choice algorithm known as BM when they have heterogeneous sophistication levels. I use two level-k models to study how students reason about the first choices reported by the others and optimally report their own first choices. My characterization allows me to compare the assignments found by BM and DA, and examine whether students can benefit from their sophistication in BM. In practice multiple cities have switched from BM to DA. The purpose of this paper is not to oppose the switch, but to provide a new perspective on understanding the effect of the switch.

Pathak and Sönmez (2013) present multiple practical school choice reforms in which the main motivation is to reduce manipulation in school choice algorithms. Pathak (2016) also emphasizes that the straightforward incentive in DA is the main reason for its re-
placement of BM in practice. However, some lab experiment and survey dataset reveal that some players attempt to manipulate DA. For example, in the experiment of Chen and Sönmez (2006) 36% of subjects attempt to manipulate DA. In the survey conducted by Rees-Jones (2015) from the participants in the 2012 National Resident Matching Program, around 5% of respondents report that they attempt to manipulate DA. So it seems that some players do not understand the strategy-proofness of DA. Ashlagi and Gonczarowski (2015) prove that DA is not obviously strategy-proof, the notion of which is easier to be understood than strategy-proofness (Li, 2015). This implies that understanding the strategy-proofness of DA may require some level of sophistication. So a natural question is how students will behave in DA if they have different abilities to understand it. It is left for future research.

References


He, Yinghua. 2014. “Gaming the boston school choice mechanism in beijing.” Manuscript, Toulouse School of Economics.


A Omitted Proofs

Proof of Proposition 1

For any $k \geq 0$, if each $i$ reports $s^k_i$ as first choice, denote the assignment found by the first round of BM by $\mu^k$. I prove by induction that $\mu^k$ is just the assignment found by round $k$ of Fast DA. Then the proposition follows.

- At $L_0$, $s^0_i$ is the most preferred school of $i$. In round 0 of Fast DA $i$ applies to his most preferred school. So it is obvious that $s^0_i = a^0_i$ and $\mu^0$ is the assignment found by round 0 of Fast DA.

- Assume that for all $r \leq k$ for some $k \geq 0$ it is true that $\mu^r$ is the assignment found by round $r$ of Fast DA. Now I consider $k + 1$. 

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If \( \mu^k(i) = s^k_i \), which means \( i \) is admitted by his reported first choice at \( Lk \) in \( \mu^k \), then \( s^k_i \) must still be \( i \)'s best obtainable school at \( Lk + 1 \). \( \mu^k(i) = s^k_i \) implies that \( i \) is assigned in round \( k \) of Fast DA. So \( s^k_i + 1 = a^k_i + 1 \). If \( \mu^k(i) = \emptyset \), \( i \) will report a school worse than \( s^k_i \) at \( Lk + 1 \). By the induction assumption the school is also the one \( i \) will apply to in round \( k + 1 \) of Fast DA. So \( s^k_i + 1 = a^k_i + 1 \).

Then \( \mu^{k+1} \) must coincide with the assignment found by round \( k + 1 \) of Fast DA.

- By induction \( \hat{s}^k_i = a^k_i \) for all \( i \) and all \( k \leq r_{FDA} \). For all \( k > r_{FDA} \), \( i \) will always report \( \hat{s}^{r_{FDA}}_i \) as first choice at \( Lk \). So \( \hat{s}^k_i = a^k_i \) for all \( i \) and all \( k > r_{FDA} \).

**Proof of Lemma 1**

I prove it by contradiction. Suppose every insufficiently sophisticated \( i \) such that \( \mu^{DA}(i) P_i \mu^{BM}_{k_i} \) reports a preference ordering \( P'_i \) such that \( \mu^{DA}(i) P'_i \mu^{BM}_{k_i} \). Let \( i_1 \) be an arbitrary such student. Since \( \mu^{DA}(i_1) P'_i \mu^{BM}_{k_i} \), \( i_1 \) must be rejected by \( \mu^{DA}(i_1) \) in some round of BM. Denote the round by \( r_1 \). Then \( \mu^{DA}(i_1) \) must admit \( q_{i_1} \) students in \( \mu^{BM}_{k_i} \), and there must exist some \( i_2 \) admitted by \( \mu^{DA}(i_1) \) in \( \mu^{BM}_{k_i} \) but \( \mu^{DA}(i_2) \neq \mu^{DA}(i_1) \).

Since \( \mu^{BM}_{k_i} \) Pareto dominates \( \mu^{DA} \), \( i_2 \) must prefer \( \mu^{DA}(i_2) \) to \( \mu^{DA}(i_1) \). Then by assumption \( \mu^{DA}(i_2) P'_{i_2} \mu^{DA}(i_1) \). Since \( i_2 \) must apply to \( \mu^{DA}(i_1) \) in a round no latter than \( r_1 \), \( i_2 \) must apply to \( \mu^{DA}(i_2) \) and be rejected in some earlier round \( r_2 \) such that \( r_2 < r_1 \). By the same argument as before, there must exist some \( i_3 \) who is admitted by \( \mu^{DA}(i_2) \) in \( \mu^{BM}_{k_i} \) but \( \mu^{DA}(i_3) \neq \mu^{DA}(i_2) \). Then \( i_3 \) must prefer \( \mu^{DA}(i_3) \) to \( \mu^{DA}(i_2) \) and is rejected by \( \mu^{DA}(i_3) \) in BM. Denote by \( r^x \) the earliest round in which some student \( i_x \) is rejected by \( \mu^{DA}(i_x) \) in BM. Then there must exist some student \( i_{x+1} \) who is admitted by \( \mu^{DA}(i_x) \) in \( \mu^{BM}_{k_i} \) but \( \mu^{DA}(i_{x+1}) \neq \mu^{DA}(i_x) \). As before, \( i_{x+1} \) must apply to \( \mu^{DA}(i_{x+1}) \) and be rejected in a round earlier than \( r^x \). But this contradicts the assumption that \( r^x \) is the earliest round in which some \( i_x \) is rejected by \( \mu^{DA}(i_x) \).

**Proof of Proposition 4**

Let \( \tilde{P} \) be the expressed preferences of \( i \) in Fast DA. Let \( \tilde{\mu} \) be a Pareto efficient assignment that Pareto dominates \( \mu^{DA} \) with respect to \( \{ \tilde{P} \}_{i \in I} \). Denote by \( \tilde{I} \equiv \{ i \in I : \tilde{\mu}(i) \tilde{P}_i \mu^{DA}(i) \} \) the set of students who are better off in \( \tilde{\mu} \) with respect to \( \{ \tilde{P} \}_{i \in I} \) than in \( \mu^{DA} \). Since every \( i \in \tilde{I} \) must apply to \( \tilde{\mu}(i) \) in some round of Fast DA, \( i \) must report \( \tilde{\mu}(i) \) as first choice at some level. Then for the level distribution in which every \( i \in \tilde{I} \) is
at the level of reporting $\tilde{\mu}(i)$ as first choice and every $j \in I \setminus \tilde{I}$ is at the level of reporting $\mu^{DA}(j)$ as first choice, the outcome of BM is just $\tilde{\mu}$. Since $\tilde{\mu}$ must Pareto dominate $\mu^{DA}$ with respect to $P_i$, the proof is finished.

**Proof of Proposition 5**

The proof is similar to that of Proposition 1. The only difference is that every $Lk$ student can see the level of every $Lk'$ student if $k' < k$. Hence in Fast DA* every $Lk'$ student cannot apply to new schools after round $k'$, and this fact is known to every $Lk$ student.

**Proof of Proposition 6**

1. By Corollary 3, a student $i$ at any level must report a school no worse than $\mu^{DA}(i)$ as first choice. So as before $\tilde{\mu}_{k_i}^{BM}$ must not be strictly Pareto dominated by $\mu^{DA}$.

2. If each student is sufficiently sophisticated, the outcome of Fast DA* must coincide with $\mu^{DA}$. So $\tilde{\mu}_{k_i}^{BM} = \mu^{DA}$.

3. If each student is insufficiently sophisticated, in the original level-k model I have shown that each $i$ must report a school strictly better than $\mu^{DA}(i)$ as first choice. In that model $i$ believes the others are $Lk_i - 1$. But in the informational level-k model if there exists some $j$ such that $k_j < k_i - 1$, then $i$ knows $j$’s level. So in $i$’s belief in the informational level-k model the market is weakly less competitive than in $i$’s belief in the original level-k model. Hence $i$ must still report a school strictly better than $\mu^{DA}(i)$ as first choice. This implies that $\tilde{\mu}_{k_i}^{BM}$ is not Pareto dominated by $\mu^{DA}$.

4. The proof of Lemma 1 implies that if $\tilde{\mu}_{k_i}^{BM}$ is Pareto dominated by $\mu^{DA}$, there must exist some positive-level $i$ who reports some $P'_i$ such that $\tilde{\mu}_{k_i}^{BM}(i) P'_i \mu^{DA}(i)$ but $\mu^{DA}(i) P_i \mu_{k_i}^{BM}(i)$. So if each positive-level $i$ reports some $P'_i$ such that $\mu^{DA}(i) P'_i s$ if $\mu^{DA}(i) P_i s$ for any $s \in S$, $\tilde{\mu}_{k_i}^{BM}$ must not be Pareto dominated by $\mu^{DA}$.

**Proof of Proposition 8**

As proved before, if $j$ is admitted by his reported first choice when being $Lk_j$, becoming a higher level does not change the outcome of BM. If $j$ is rejected by his reported first
choice when being $Lk_j$, let $s$ be the school that finally admits $j$ in BM. Then $s$ must have empty seats after the first round of BM. In other words, $j$ is unassigned in $\mu_{k_i}^{FDA^*}$ and $s$ has empty seats in $\mu_{k_i}^{FDA^*}$. Now suppose $j$ becomes $Lk_j'$ for any $k_j' > k_j$. To obtain the outcome of the first round of BM I study the outcome of Fast DA*. In Fast DA*, $j$ will apply to some new schools after round $k_j$. Let the sequence of these new schools be $\{s_1, \cdots, s_v\}$. Since $k_j = \bar{k}_N$, no student in $N \backslash \{j\}$ will apply to new schools after round $k_j$ of Fast DA*. Hence to obtain the new outcome of Fast DA* I start with the old outcome $\mu_{k_i}^{FDA^*}$ and let $j$ apply to the schools in the sequence one by one. When $j$ applies to a school in the sequence, $j$ is either rejected immediately, or is tentatively accepted and possibly induces a rejection-application chain.

Let $s_a$ be the first school in the sequence that accepts $j$ tentatively. Since $s$ has empty seats, $s$ will accept $j$ immediately if $j$ applies to $s$. So $s_a$ must be weakly better than $s$. If $s_a$ finally accepts $j$, which means $j$ is weakly better off, I finish the proof. If $s_a$ finally rejects $j$, then $j$ must induce a rejection-application chain in which a student with a higher priority than $j$ at $s_a$ applies to $s_a$ and replaces $j$. Since any student in $N \backslash \{j\}$ cannot apply to a new school if being rejected, the chain must not involve any student in $N \backslash \{j\}$. The chain must neither involve any school with empty seats. So after $j$ being rejected by $s_a$, the only change in the outcome of Fast DA* is that some quasi-rational students exchange their seats. The set of unassigned students and the set of empty seats are same as before. Then I can repeat the above argument for $s_{a+1}$ and all following schools. If $j$ is rejected by all schools in the sequence, those schools must be strictly better than $s$, and $s_v$ is just the first choice reported by $j$ when being $Lk_j'$. So all schools weakly better than $s_v$ must be exhausted in the new outcome of Fast DA*, and the set of unassigned students and the set of empty seats in the new outcome of Fast DA* are same as before. Note that all unassigned students will report the same preferences as before since they cannot know the level change of $j$. Hence if $j$ uses a strategy satisfying the worse-rank condition, $j$ will apply to every school worse than $s_v$ in the same round of BM as before, and every other unassigned student will apply to the same school in the same round of BM as he did before. So $j$ must still be admitted by $s$. Hence, I finish the proof.
Results of Pathak and Sönmez (2008) are Corollaries

I prove that all results of Pathak and Sönmez (2008) are either implied by mine or can be proved easily through my method. For any $P_I$, if any nonempty $N \subseteq I$ is the set of naive students and any nonempty $M = I \setminus N$ is the set of rational students, I prove that the set of NE outcomes of BM can be found by the following two-step procedure.

- Construct an artificial economy $(\{P^1_j\}_{j \in N}, \{P_\ell\}_{\ell \in M})$ where $P^1_j$ only lists the most preferred school of $j$. Let $\mathcal{M}_1$ be the set of stable assignments in this economy.

- For each $\mu \in \mathcal{M}_1$, finalize the assignments of all assigned students, then run BM for unassigned students using their true preferences. Denote the assignment found in this way by $f(\mu)$. Define $\mathcal{M}_2 \equiv \{f(\mu) : \mu \in \mathcal{M}_1\}$.

Proposition 10. $\mathcal{M}_2$ is the set of NE outcomes of BM when $N$ is naive and $M$ is rational.

Proof. By Ergin and Sönmez (2006), $\mathcal{M}_1$ is the set of NE outcomes of BM in the artificial economy if all students are rational. However, since each $j \in N$ is allowed to list only one school, $j$ cannot manipulate BM in the artificial economy. For each $\mu \in \mathcal{M}_1$, the students in $M$ obtain the same assignments in $\mu$ and $f(\mu)$. So it must be a NE of BM that each $j \in N$ reports $P_j$ and each $\ell \in M$ reports $\mu(\ell)$ as first choice in $P_I$. Hence, $f(\mu)$ is a NE outcome of BM in $P_I$.

Conversely, for each NE outcome $\mu$ in $P_I$, it must be a NE that each $j \in N$ reports $P_j$ and each $\ell \in M$ reports $\mu(\ell)$ as first choice. Then in the artificial economy it must still be a NE that each $j \in N$ reports $P^1_j$ and each $\ell \in M$ reports $\mu(\ell)$ as first choice. Note that the corresponding outcome of BM in the artificial economy must be $f^{-1}(\mu)$. Hence, I finish the proof.

If I denote the student-optimal stable assignment in the artificial economy by $\mu^*$, then $f(\mu^*)$ must be the student-optimal stable assignment in $P_I$. So $f(\mu^*) = \mu^\text{DA}$. In the informational level-k model of BM, if $k_N = 0$, then all students in $N$ can only apply to their most preferred schools in Fast DA*. So the outcome of Fast DA* is just $\mu^*$. Note that this is the assignment found by the first round of BM. So running the remaining
procedures of BM is equivalent to the second step of the above two-stage procedure. So
the outcome of BM is just $f(\mu^*) = \mu^{DA}$. This proves Proposition 9. So the comparative
statics results of Pathak and Sönmez are implied by Proposition 5 and Proposition 6.

**Corollary 7.** (Propositions 3 and 4 of Pathak and Sönmez (2008)) Rational students are
weakly better off in the student-optimal NE outcome of BM than in the outcome of DA. A
naive student weakly benefits from becoming rational and all rational students weakly
suffer.

Pathak and Sönmez also prove that each $j \in N$ obtains the same assignment in all NE
outcomes of BM in $P_1$. I show that it is straightforwardly implied by the rural hospital
theorem (Roth, 1986). Specifically, for each $\mu \in \mathcal{M}_1$, all students in $M$ must be assigned
in $\mu$. Each $j \in N$ is either assigned to his most preferred school or unassigned in $\mu$. By
the rural hospital theorem, if $j$ is assigned in one stable assignment, $j$ must be assigned
in all stable assignments. Hence, if $j$ is assigned in one $\mu$, $j$ must be assigned to his most
preferred school in all $\mu \in \mathcal{M}_1$ as well as in $\mathcal{M}_2$. On the other hand, by the rural hospital
theorem each school must admit the same number of students in all $\mu \in \mathcal{M}_1$. Hence, the
number of unassigned students and the number of empty seats at each school are same in
all $\mu \in \mathcal{M}_1$. Then at the second step of the above two-stage procedure each unassigned
$j \in N$ must obtain the same assignment in all $\mu \in \mathcal{M}_2$.

**Corollary 8.** (Proposition 2 of Pathak and Sönmez (2008)) Each $j \in N$ is admitted by
the same school in all NE outcomes of BM.

### C Additional Simulation Results

Figure 2 reports the summary of the simulation results for the informational level-k model.
The figure is very close to the original level-k model in Figure 1. To check the robustness
of my results I also consider unbalanced markets in which there are more students than
schools. To model it I let some school be the worst one in all students’ preferences. Then
Figure 3 reports the summary of the simulation results for the original level-k model in
unbalanced markets. The results for the informational level-k model are very similar and
omitted.
D A Level-k Model of Constrained DA

Let \( c < |S| \) be the constraint on the length of preference orderings that students can report. I provide an original level-k model of constrained DA by assuming that \( L_0 \) students report their top \( c \) choices and positive-level students use topping strategies.\(^\text{15}\)

Formally, let \( P^c \) be the truncated version of any \( P \in \mathcal{P} \) that ranks the top \( c \) choices. When each \( i \) reports \( P^c_i \), denote the outcome of DA by \( \mu^c \). If \( \mu^c(i) \neq \emptyset \), then \( \mu^c(i) \) must be weakly better than \( \mu^{DA}(i) \). If all students are assigned in \( \mu^c \), then \( \mu^c = \mu^{DA} \). Let \( s^k_i \) be the first choice reported by each \( i \) at \( L_k \), then I have the following result.

**Proposition 11.** For any \( P_i, s^k_i R_i s^{k+1}_i R_i \mu^{DA}(i) \) for all \( i \in I \) and all \( k \geq 0 \). There

\(^\text{15}\)That is, they report their best obtained schools as first choices and report the true preference ordering of the remaining \( c - 1 \) top choices.
exists some finite $r_{DA}^i \geq 0$ for each $i$ such that $s_{k}^i = \mu_{DA}^i(i)$ for all $k \geq r_{DA}^i$.

**Proof.** At $L0$ each $i$ reports $P_c^i$. So $s_{0}^i$ is the most preferred school of $i$. Denote the outcome of DA if all students are $L0$ by $\mu^0$. It is obvious that $\mu^0 = \mu^c$. If any school $s$ admits $q_s$ students in $\mu^0$, denote the priority rank of the lowest-priority admitted student by $z_{0}^s$. Otherwise, define $z_{0}^s \equiv |I|$. So $z_{0}^s$ is the threshold of entering $s$ in $\mu^0$. Let $z_{DA}^s$ be the similar threshold in $\mu_{DA}^s$, then it is obvious that $z_{0}^s \geq z_{DA}^s$ for all $s$.

At $L1$, for each $i$, if $\mu^0(i) \neq \emptyset$, then $\mu^0(i)$ is the best obtainable school for $i$. Hence, $s_{1}^i = \mu^0(i)$. If $\mu^0(i) = \emptyset$, then $i$ will report a new best obtainable school $s_{1}^i$ as first choice. Since $z_{0}^s \geq z_{DA}^s$ for all $s$, $\mu_{DA}^i(i)$ must be obtainable for $i$. So $s_{1}^i$ must be weakly better than $\mu_{DA}^i(i)$. Denote the outcome of DA if all students are $L1$ by $\mu^1$. Denote the threshold of entering each $s$ in $\mu^1$ by $z_{s}^1$. Then $z_{s}^1 \leq z_{0}^s$ for all $s$, which means all
thresholds are weakly higher. By using topping strategies each $i$ is either unassigned in $\mu^1$ or admitted by a school weakly better than $\mu^{DA}(i)$. So $z^{DA}_s \leq z^1_s$ for all $s$.

At $Lk$ for any $k \geq 2$, suppose it is true that $s^{k-1}_i R_i s^k_i R_i \mu^{DA}(i)$ and $z^{DA}_s \leq z^k_s \leq z^{k-1}_s$ for all $i$, all $s$ and all $k' < k$. Then for each $i$, any $s$ better than $s^{k-1}_i$ must be unobtainable for $i$. If $\mu^{k-1}(i) \neq \emptyset$, then $\mu^{k-1}(i)$ is the best obtainable school for $i$. If $\mu^{k-1}(i) = \emptyset$, $i$ will report a new first choice at $Lk$. But the school must be weakly better than $\mu^{DA}(i)$ since $z^{DA}_s \leq z^{k-1}_s$ for all $s$. Hence, it is still true that $s^{k-1}_i R_i s^k_i R_i \mu^{DA}(i)$ and $z^{DA}_s \leq z^k_s \leq z^{k-1}_s$ for all $i$ and all $s$. Then by induction, $s^k_i R_i s^{k+1}_i R_i \mu^{DA}(i)$ for all $i \in I$ and all $k \geq 0$.

When $k$ is high enough, all students at $Lk$ must be assigned in $\mu^k$. Then $\mu^k$ must weakly Pareto dominate $\mu^{DA}$. Since $\mu^k(i)$ is the best obtainable school for each $i$ at $L(k+1)$, $\mu^k$ must be stable. So $\mu^k = \mu^{DA}$. Hence, there exists some finite $r^{DA}_i \geq 0$ for each $i$ such that $s^k_i = \mu^{DA}(i)$ for all $k \geq r^{DA}_i$.

So the above original level-k model of constrained DA looks like that of constrained BM. If all students have high enough levels, the outcomes of both BM and DA are $\mu^{DA}$. If some students have low levels, the comparison between BM and DA is ambiguous. Therefore, I do simulations to compare them by choosing $c = 5$ and letting positive-level students use topping strategies. Figure 4 shows that DA is more efficient than BM. Specifically, there are more students who prefer the assignments in DA than those who prefer the assignments in BM. Sophistication level is also significantly correlated with a student’s welfare in both BM and DA. However, as discussed before, the level-k model of constrained DA depends on the selection of students’ best strategies at positive levels. By using topping strategies positive-level students may report non-truthful rankings of the schools they report, which are proved to be weakly dominated by truthful rankings of the schools they report (Calsamiglia, Haeringer and Klijn, 2010). So to what an extent the above level-k model of DA and the simulation results are credible is still a question.

E Correction of Proposition 1 of Abdulkadiroğlu, Che and Yasuda (2011)

Abdulkadiroğlu, Che and Yasuda (2011) prove their Proposition 1 in their on-line ap-
Figure 4: Compare BM and DA in constrained school choice.

Appendix, which states that if students have common preferences in the strict priorities and complete information environment and are either naive or rational, then if any naive student becoming rational, in the unique NE outcome of BM all the other naive students will be weakly worse off. I show that this statement is actually incorrect through the following example.

Example 2. There are four schools \( \{s_1, s_2, s_3, s_4\} \) and four students \( \{i_1, i_2, i_3, i_4\} \). Each school has only one seat. Students have the common preference ordering \( s_1 \succ s_2 \succ s_3 \succ s_4 \). The priority rankings are shown below. Suppose all students are naive, then they report true preferences. Now if \( i_2 \) becomes rational, he will report \( s_2 \) as first choice since \( s_1 \) must be obtained by \( i_1 \). The outcomes of BM are shown respectively below. It is easy to see that \( i_2, i_4 \) are better off, \( i_1 \) remains same, and \( i_3 \) is worse off.

Now I provide a detailed analysis of the statement using my method in Appendix B. When some naive student \( j \) becomes rational, by Proposition 9 it is equivalent to the situation that in the informational level-k model of BM \( k_N = 0 \), but some \( j \in N \) becomes quasi-rational. Suppose there are \( m \) schools and the common preferences are \( s_1 \succ s_2 \succ \cdots \succ s_m \). As proved before, if \( j \) is assigned in Fast DA\(^*\) when being \( L0, \)
becoming quasi-rational does not change the outcome of BM. If \( j \) is unassigned in Fast DA\(^*\) when being \( L0 \), let \( \{s_v, s_{v+1}, \ldots, s_m\} \) be the set of schools with empty seats in the outcome of Fast DA\(^*\). Let the number of empty seats of \( s_v \) be \( e_{s_v} \). Since students have common preferences, the number of empty seats of each \( s_a \) with \( a > v \) must be \( q_{s_a} \). Let \( s_u \) be the school that admits \( j \) in the outcome of BM. So \( u \in [v, m] \).

By becoming quasi-rational \( j \) must be assigned in Fast DA\(^*\). So from the second round of BM on there will be one fewer empty seat and one fewer unassigned student than before. Since students have common preferences, the missing empty seat must belong to \( s_v \). Hence, \( s_v \) has \( e_{s_v} - 1 \) empty seats now. So some student \( j_1 \) who was admitted by \( s_v \) in the second round of BM before will be rejected by \( s_v \) now. Then \( j_1 \) will apply to \( s_{v+1} \), and either be rejected or replace another student who was admitted by \( s_{v+1} \) before. Repeating this argument there must be exactly one student \( j_a \) who was admitted by some school in \( \{s_v, \ldots, s_{u-1}\} \) but now is rejected. Moreover, all the students who were admitted by the schools in \( \{s_v, \ldots, s_{u-1}\} \) before are weakly worse off and some is strictly worse off (e.g., \( j_1 \) and \( j_a \)). \( j_a \) will apply to \( s_u \). Since \( j \) is assigned in the first round, compared to the before situation \( j_a \) essentially replaces \( j \) among those who applied to \( s_u \) before. Let \( j_b \) be the highest-priority student at \( s_u \) who was rejected by \( s_u \) before. If \( j_b \) has a higher priority than \( j_a \), then \( j_b \) will be admitted by \( s_u \) and be strictly better off (this is what happens in the above example). Then \( j_a \) will apply to \( s_{u+1} \) and essentially replace \( j_b \) among those who applied to \( s_{u+1} \) before. On the other hand, if \( j_b \) has a lower priority than \( j_a \) at \( s_u \), then \( j_a \) is admitted by \( s_u \). Then the assignments of all naive students who were admitted by the schools in \( \{s_{u+1}, \ldots, s_m\} \) do not change.