Abstract

I study how agents of different sophistication levels reason and behave in a centralized system that matches students with public schools. I use two level-k models to study the strategies used by students in the popular Boston mechanism (BM), and compare the outcome of BM against that of another popular mechanism known as Deferred Acceptance (DA). I find that the level-k reasoning process in BM is analogous to the procedure of DA. This fact implies that the outcome of BM is no less efficient than that of DA for all possible sophistication levels of students. I examine whether a student benefits from his sophistication in BM. I find that the answer depends on the student’s level and his belief of others’ levels. I also simulate the outcomes of BM and DA by randomly generating the levels of students. Simulation results are consistent with recent empirical estimations.

Keywords: School choice, Boston mechanism, Deferred Acceptance, Level-k model

JEL Classification: C78, D61, D78, I21, I28
1 Introduction

A main objective of economists is to understand the operation of markets and propose methods to improve them. With the development of game theory and experimental economics, many economists are involved in the practice of designing markets, such as designing auctions to sell public resources, on-line search platforms to sell advertisements, centralized systems to allocate donated organs, and clearinghouses to clear labor markets. In this paper I study the design of a centralized school choice system to match students with public schools. I want to answer the following questions: (i) how do students of heterogeneous sophistication behave in popular matching mechanisms, and (ii) how does the answer affect the evaluation of these mechanisms? It is well known in economics that people are often boundedly rational and have heterogeneous sophistication, but this fact has not been explored well in matching theory.¹

Two mechanisms are widely used in the practice of school choice: the Boston mechanism (BM) and the student-proposing deferred acceptance (DA). BM and its variants are used by many cities, while DA was proposed by Gale and Shapley (1962) and then adapted by Abdulkadiro˘ glu and Sönmez (2003) to school choice. The biggest difference between the two mechanisms is that truth-telling is a dominant strategy in DA, while BM is a priority mechanism (Roth, 1991), which is manipulable by students.² After being recommended by economists, DA has replaced BM in many cities in the past decade (Pathak and Sönmez, 2013). In this paper I study how the existence of heterogeneous sophistication affects our understanding of the two mechanisms.

To model heterogeneous sophistication I use level-k models. The model was first proposed by Stahl and Wilson (1994, 1995) and Nagel (1995), then developed by Ho, Camerer and Weigelt (1998); Costa-Gomes, Crawford and Broseta (2001); Costa-Gomes and Crawford (2006); Crawford and Iriberri (2007a,b); Arad and Rubinstein (2012), and many others. Compared with other non-equilibrium models, the level-k model is tractable and has effective explanatory power. When applying the model to school choice, students have discrete sophistication levels, which are their depths of strategic reasoning. If a

¹Bade (2016) is the only paper I am aware of that discusses matching mechanisms for boundedly rational agents. She studies the allocation of indivisible objects without priorities and examines whether the Pareto optimality of popular mechanisms is robust to any deviation from rationality.

²Public schools are regulated to follow given admission policies, so they are not strategic.
student’s level is $k$ for any $k \geq 0$, he is said to be level-$k$, denoted by $Lk$ for short. An $L0$ student is naive and does not reason strategically, so he reports true preferences. An $Lk$ student for any $k > 0$ engages in strategic reasoning based on his belief about others’ levels. I consider two extreme settings of his belief, which correspond to two level-$k$ models. In the first setting an $Lk$ student believes all others are $Lk - 1$. This setting is commonly used in the literature, so I refer to the corresponding model as the *original level-$k$ model*. In the second setting, for any realized distribution of students’ levels, an $Lk$ student knows the true levels of others whose levels are lower than $k$, and believes the remaining others are $Lk - 1$. By the spirit of the level-$k$ model an $Lk$ student cannot believe any other student’s level is higher than $k - 1$. So in the second setting an $Lk$ student has the most information he can have about others’ levels, and I refer to the corresponding model as the *informational level-$k$ model*.

In DA truth-telling is a dominant strategy, so students do not engage in strategic reasoning even though they have positive levels. Therefore, my main task is to analyze the strategies used by students in BM. To study the question in the most transparent environment, I assume complete information. That is, preference and priority profiles are commonly known to students. Then a student at a positive level infers others’ strategies based on his belief, and chooses his optimal strategy. I do not make any assumption about the distribution of students’ levels, and characterize the optimal strategy each student uses at each possible level. Since first choice plays the most important role in determining a student’s assignment in BM, I assume that when a positive-level student chooses his optimal strategy, he optimally reports his first choice. For most of my results I do not assume how positive-level students report their whole preference orderings. I summarize my findings below.

First, in both level-$k$ models the iterated reasoning processes in BM are analogous to the procedure of DA.\textsuperscript{3} Specifically, when a positive-level student reasons strategically in BM, it is as if he runs DA in his mind to decide the best first choice he should report, and when his level is higher, he runs more rounds of DA in his mind. To illustrate, let me take an $L1$ student as an example. By believing others are $L0$, he infers that others report their most preferred schools as first choice. Then he checks whether he can obtain his most preferred school by reporting it as first choice, and if not, which school is his

\textsuperscript{3}Formally, the iterated reasoning processes coincide with the procedures of two variants of DA.
best first choice. It is as if he runs the first round of DA in his mind to determine whether he will be admitted by his most preferred school, and if not, which school he will apply to in the second round of DA.

Second, I show that the outcome of BM in both level-k models is not (strictly) Pareto dominated by that of DA. Specifically, the above analogy implies that in both level-k models of BM a student reports a weakly worse school as first choice at a higher level, but the school is never worse than his DA assignment. In the original level-k model of BM, each student has an individual level threshold above which he reports his DA assignment as first choice and obtains it for sure. In both level-k models of BM, if each student’s level is above his individual threshold, the outcome of BM coincides with that of DA; if each student’s level is below his individual threshold, the outcome of BM is not Pareto dominated by that of DA. For other level distributions I prove that the outcome of BM is never strictly Pareto dominated by that of DA, and under a mild assumption on the strategies of positive-level students, the outcome of BM is never Pareto dominated by that of DA.

Last, I examine the relation between a student’s sophistication level and his welfare in both level-k models of BM. In particular, I examine whether a student obtains a better assignment in BM at a higher level. In the original level-k model the answer is uncertain. It is because the belief of an $L_k$ student for any $k > 0$ about others’ levels is generally incorrect, and the welfare effect of this belief is ambiguous. However, in the informational level-k model the answer is positive if the student’s level is either the highest in the student population or is above his threshold. In particular, I show that his assignment must be weakly better than his DA assignment.

To summarize, although the two level-k models make extreme assumptions about the beliefs of positive-level students, in both models I show that the iterated reasoning process in BM can be understood through the procedure of DA. Then I prove that BM is no less efficient than DA. Hence, in any other level-k model that makes a moderate assumption about the beliefs of positive-level students, I believe similar results will still hold. However, the contrast between the two models suggests that a correct belief about others’ levels is crucial for a student to benefit from his sophistication in BM. When a student has an incorrect belief, his strategy can be too optimistic or pessimistic. Nevertheless, many experiments have shown that people often have low levels, and in practice it is hard for
students to estimate others’ levels very well. So my results imply that in practice it is hard for a sophisticated student to benefit from his sophistication in BM.

To quantify the difference between the outcomes of BM and DA, I simulate them by randomly generating the levels of students. Simulation results show that neither mechanism clearly dominates the other. Specifically, the percentage of students who obtain better assignments in BM is often higher than the percentage of students who obtain better assignments in DA, and both percentages are significantly above zero. The average welfare of students is slightly higher in BM than in DA. These features are consistent with recent empirical studies by He (2014) and Calsamiglia, Fu and Güell (2015). He estimates that if BM is replaced by DA in Beijing, an average student would suffer a significant utility loss and the number of worse off students is significantly more than that of better off students. Calsamiglia, Fu and Güell estimate that the replacement of BM with DA in Barcelona would benefit fewer than 10% of students but hurt 28% of students, and an average student would lose by an amount equivalent to 60 euro. My simulation results also show that in both level-k models of BM, the correlation coefficient between a student’s welfare and his level is positive. So although my theoretical analysis implies that it is hard for a student to benefit from his sophistication in BM, my simulation suggests that it can happen statistically. This feature is consistent with He’s estimation that an average sophisticated student would suffer from the replacement of BM with DA in Beijing.

In the literature, the arguments to support the replacement of BM with DA can be roughly summarized as follows. First, the straightforward incentive in DA motivates cities to switch their mechanisms (Pathak and Sönmez, 2013; Pathak, 2016). Second, the outcome of BM is often Pareto dominated by that of DA when students manipulate BM (Ergin and Sönmez, 2006). Last, sophisticated students can take advantage of naive students to achieve better manipulation in BM, while DA excludes this possibility. In the proofs of the latter two arguments students are assumed to be either rational or naive. This dichotomous assumption can be seen as a special case of my level-k models if students are either $L_0$ or have high levels. I prove that the results of Pathak and Sönmez (2008) are corollaries of mine in the informational level-k model. The merit of my level-k models is that they provide richer settings to examine the effect of heterogeneous sophistication and the validity of the above arguments.
I provide an example to illustrate my main idea in Section 2. In Section 3 I give formal definitions of school choice problems and the two mechanisms. I analyze the original level-k model in Section 4 and the informational level-k model in Section 5. I include discussion in Section 6 and present simulations in Section 7. I discuss extensions in Section 8 and related literature in Section 9. Section 10 concludes. The appendix includes omitted proofs and additional results.

2 An Example

Consider a school choice problem with three students $1, 2, 3$ and three schools $a, b, c$. Each school has only one seat. I will use $i$ to denote a typical student and $s$ to denote a typical school. The following table shows the preference ordering $P_i$ of each $i$ and the priority ranking $\pi_s$ of each $s$.

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$\pi_a$</th>
<th>$\pi_b$</th>
<th>$\pi_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$a$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

2.1 The Procedures of BM and DA

In both BM and DA, after students report preferences, their applications are sent to schools according to their reported preference orderings. When schools receive applications, they admit applicants according to their priority rankings. In each round of BM only schools with empty seats can accept new applicants, while in each round of DA each school will compare the students it admitted earlier with new applicants in this round, and admit them according to its priority ranking. Table 1 shows the procedures of BM and DA when all students report true preferences. In the table $i \rightarrow s$ means that the application of $i$ is sent to $s$. The difference between BM and DA can be illustrated well by round 2. In BM, $2$ is rejected by $b$ since $b$ has no empty seat, but in DA, $b$ compares $2$ and $3$, and admits $2$.

The outcomes of BM and DA are shown in Table 2. In BM, if a student ranks a school higher in his reported preferences, his application will be sent to the school earlier. So
students are most likely to manipulate BM by misreporting first choice. In this example 2 can manipulate BM by reporting $b$ as first choice.

<table>
<thead>
<tr>
<th></th>
<th>BM</th>
<th>DA</th>
</tr>
</thead>
<tbody>
<tr>
<td>round 1</td>
<td>$1 \rightarrow a, 2 \rightarrow a, 3 \rightarrow b$</td>
<td>$1 \rightarrow a, 2 \rightarrow a, 3 \rightarrow b$</td>
</tr>
<tr>
<td>round 2</td>
<td>$2 \rightarrow b$</td>
<td>$2 \rightarrow b$</td>
</tr>
<tr>
<td>round 3</td>
<td>$2 \rightarrow c$</td>
<td>$3 \rightarrow a$</td>
</tr>
<tr>
<td>round 4</td>
<td>$1 \rightarrow b$</td>
<td></td>
</tr>
<tr>
<td>round 5</td>
<td></td>
<td>$2 \rightarrow c$</td>
</tr>
</tbody>
</table>

Table 1: Procedures of BM and DA

2.2 The Original Level-k Model of BM

I use the example to illustrate the iterated reasoning process in the original level-k model of BM. I assume that when a positive-level student attempts to manipulate BM, he misreports his first choice. In the following I discuss the first choice reported by each student at each positive level, and use $i \rightarrow s$ to denote that $i$ reports $s$ as first choice.

- **L0**: Each $i$ reports true preferences. So,

  $$1 \rightarrow a, 2 \rightarrow a, 3 \rightarrow b$$

- **L1**: Each $i$ believes others report true preferences. 1 still reports $a$ as first choice and 3 still reports $b$ as first choice. But 2 reports $b$ as first choice since he believes that if he reports $a$ as first choice, he would not obtain $a$ but would also lose $b$. So,

  $$1 \rightarrow a, 2 \rightarrow b, 3 \rightarrow b$$
• $L2$: Each $i$ believes others are $L1$. By repeating the $L1$ reasoning, $i$ infers the first choices reported by others. Then 1 still reports $a$ as first choice and 2 still reports $b$ as first choice. But 3 reports $a$ as first choice since he believes that if he reports $b$ as first choice, he would not obtain $b$ but would also lose $a$. So,

$$1 \rightarrow a, \ 2 \rightarrow b, \ 3 \rightarrow a$$

• $L3$: Each $i$ believes the others are $L2$ and infers their reported first choices. Then 2 still reports $b$ as first choice and 3 still reports $a$ as first choice. But 1 reports $b$ as first choice since he believes if he reports $a$ as first choice, he would not obtain $a$ but would also lose $b$. So,

$$1 \rightarrow b, \ 2 \rightarrow b, \ 3 \rightarrow a$$

• $L4$: Each $i$ believes the others are $L3$ and infers their reported first choices. Then 1 still reports $b$ as first choice and 3 still reports $a$ as first choice. But 2 reports $c$ as first choice since he would not obtain $a$ or $b$ if he reports either as first choice. So,

$$1 \rightarrow b, \ 2 \rightarrow c, \ 3 \rightarrow a$$

• $Lk \ (k > 4)$: Each $i$ believes the others are $Lk - 1$ and infers their reported first choices. Then all students report the same first choices as they do at $L4$.

It is easy to see that the above reasoning process coincides with the procedure of DA. In particular, the $L0$ reasoning coincides with round 1 of DA in which all students apply to their most preferred schools; the $L1$ reasoning coincides with round 2 of DA in which students rejected in round 1 apply to new schools. In general, the iterated reasoning process coincides with the procedure of a variant of DA that I call Fast DA. Fast DA runs faster than DA, but they always find the same outcome. In this example Fast DA and DA have same procedures.

**Compare BM and DA** If denoting the outcome of DA by $\mu^{DA}$, in the above reasoning process any $i$ at any $Lk$ reports a school no worse than $\mu^{DA}(i)$ as first choice, and there exists a threshold level above which $i$ reports $\mu^{DA}(i)$ as first choice. This observation implies that for any level distribution of students, the outcome of BM is never strictly
Pareto dominated by $\mu^{DA}$, and when each $i$ reports $\mu^{DA}(i)$ as first choice, the outcome of BM is just $\mu^{DA}$. In this example, the outcome of BM can Pareto dominate $\mu^{DA}$ for some level distribution. For example, if 1, 3 are $L0$ and 2 is $L4$, then 1, 3 obtain their most preferred schools, which are better than their DA assignments, and 2 obtains his DA assignment.

**The benefit of sophistication** To investigate whether a student’s welfare in BM is correlated with his level, I increase his level to see how the welfare of the student and others changes accordingly. In the original level-k model I show that the welfare change is ambiguous in general. In particular, the student may not be better off, while others may not be worse off. In this example, suppose 2, 3 are $L1$, and students at positive levels report true preference orderings of the schools below their reported first choices. Suppose 1’s level is increased from $L0$ to $L3$, then the outcomes of BM are shown below. It is easy to see that 1, 2 are worse off, while 3 is better off.

<table>
<thead>
<tr>
<th>1 2 3</th>
<th>1 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a b c</td>
<td>b c a</td>
</tr>
</tbody>
</table>

(a) 1 is $L0$ and 2, 3 are $L1$  
(b) 1 is $L3$ and 2, 3 are $L1$

### 2.3 The Informational Level-k Model of BM

In the informational level-k model, an $Lk$ student for any $k > 0$ knows the levels of those whose levels are lower than $k$. So his strategy depends on others’ levels. To illustrate the level-k reasoning in this model through the example, I consider a level distribution that 1 is $L1$, 2 is $L4$, and 3 is $L2$. I show the reasoning process in Table 4. In particular, because 2 knows others’ levels, he reports $b$ as first choice at $L4$, which is different from his report $c$ in the original level-k model.

The reasoning process in this model coincides with the procedure of another variant of DA. This implies that the efficiency comparison between BM and DA in the original level-k model still holds. However, this model is different from the previous one in that when a student’s level is high enough, he always obtains his reported first choice. Then I prove that his assignment in BM must be weakly better than his DA assignment, and he does not like any student of a lower level becoming more sophisticated. In this example
Table 4

<table>
<thead>
<tr>
<th></th>
<th>1 → a</th>
<th>2 → a</th>
<th>3 → b</th>
</tr>
</thead>
<tbody>
<tr>
<td>L0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L1</td>
<td>1 → a</td>
<td>2 → b</td>
<td>3 → b</td>
</tr>
<tr>
<td>L2</td>
<td></td>
<td>2 → b</td>
<td>3 → a</td>
</tr>
<tr>
<td>L3</td>
<td>2 → b</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L4</td>
<td>2 → b</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2 has the highest level among all students. He correctly infers others’ strategies and obtains his reported first choice \( b \), which is better than his DA assignment. Now suppose 1 becomes \( L3 \), then 1 will report \( b \) as first choice, and 2 will respond to it by reporting \( c \) as first choice. Then 1 will obtain \( b \) and be better off, while 2 will obtain \( c \) and be worse off.

3 The Model

A school choice problem consists of the following elements:

- a finite set of students \( I \);
- a finite set of schools \( S \);
- a capacity vector \( Q = \{q_s\}_{s \in S} \) where \( q_s \) is the number of seats at school \( s \);
- a priority profile of schools \( \Pi_S = \{\pi_s\}_{s \in S} \) where \( \pi_s \) is the strict priority ranking of school \( s \) over students;
- a preference profile of students \( P_I = \{P_i\}_{i \in I} \) where \( P_i \) is the strict preference ordering of student \( i \) over schools.

I assume \( \sum_{s \in S} q_s = |I| \). That is, there are enough seats to admit all students. This assumption accommodates two cases. In the first case the law in many countries requires each student attend a public school, so it is natural to assume enough seats. In the second case if students have outside options (private schools or studying at home), some school in \( S \) can denote such an outside option. I use \( R_i \) to denote the weak preference ordering associated with \( P_i \), use \( \mathcal{P} \) to denote the set of all strict preference orderings, and use \( \mathcal{O} \)
to denote the set of all school choice problems. Throughout the paper I fix $I$, $S$ and $Q$.
So a school choice problem can be simply denoted by $P_I$.

A matching between students and schools is a function $\mu : I \rightarrow S$ such that $|\mu^{-1}(s)| \leq q_s$ for all $s \in S$. Here $\mu(i)$ is the assignment of each student $i$ and $\mu^{-1}(s)$ is the set of students admitted by each school $s$. I denote the set of all matchings by $\mathcal{M}$. A student $i$ justified envies another student $j$ in a matching $\mu$ if $\mu(j)P_i\mu(i)$ and $i\pi_{\mu(j)}j$. That is, $i$ has a higher priority than $j$ at school $\mu(j)$ but $i$ is assigned to a school worse than $\mu(j)$. A matching $\mu$ is wasteful if $|\mu^{-1}(s)| < q_s$ and $sP_i\mu(i)$ for some $s$ and some $i$. That is, $s$ has empty seats and $i$ prefers $s$ to his assignment. A matching is stable if it does not contain justified envies and is not wasteful.

A matching $\mu$ Pareto dominates another matching $\mu'$ if $\mu(i)R_i\mu'(i)$ for all $i \in I$, and $\mu(j)P_j\mu'(j)$ for some $j \in I$. If $\mu(i)P_i\mu'(i)$ for all $i \in I$, $\mu$ strictly Pareto dominates $\mu'$. A matching is Pareto efficient if it is not Pareto dominated by any other matching. There exists a stable matching that Pareto dominates any other stable matching. It is called student-optimal stable and denoted by $\mu^{DA}$.

A school choice mechanism is a function $\psi : \mathcal{O} \rightarrow \mathcal{M}$ such that $\psi(P_I)$ is the matching found for $P_I$. $\psi$ is Pareto efficient or stable if it always finds a Pareto efficient or stable matching. $\psi$ is strategy-proof if all students are willing to report true preferences. Formally, $\psi(P_I)(i)R_i\psi\{P'_i,P_{-i}\}(i)$ for all $i \in I$, all $P_I \in \mathcal{P}^{[I]}$ and all $P'_i \in \mathcal{P}$.

BM and DA are two popular school choice mechanisms. Their difference has been illustrated by the example in Section 2. In the following I present their formal definitions.

**The Procedures of BM and DA**

**Round 1:** Each student applies to the first choice in his reported preference ordering. Each school admits its applicants one by one according to its priority ranking until all seats are occupied or all applicants are admitted. Remaining applicants, if any, are rejected. If after this round all students are admitted, stop the procedure.

**Round $r \geq 2$:** Each unmatched student applies to the next school in his reported preference ordering.

- In BM, each school with empty seats admits its applicants one by one according to its priority ranking until all seats are occupied or all applicants are admitted.
Remaining applicants, if any, are rejected. If after this round all students are admitted, stop the procedure.

- In DA, each school receiving new applications admits students one by one from its earlier admitted students and new applicants according to its priority ranking until all seats are occupied or all students are admitted. Remaining students, if any, are rejected. If after this round all students are admitted, stop the procedure.

4 The Original Level-k Model of BM

In the original level-k model of BM an $Lk$ student $i$ for any $k > 0$ believes the others are $Lk - 1$. He infers the strategies of the others in the complete information environment, then chooses his best strategy. For any given $P_I$, I say a school $s$ is obtainable to $i$ if $i$ believes he can obtain $s$ by reporting some preference ordering. So any preference ordering that lets $i$ obtain his best obtainable school is a best strategy for him. Since in BM students are most likely to misreport first choice, I assume that $i$ reports his best obtainable school as first choice.

**Best first choice assumption:** An $Lk$ student for any $k > 0$ reports his best obtainable school as first choice.

I use $s^k_i$ to denote the school each $i$ reports as first choice at any $Lk$. In the following I define a variant of DA, and prove that the reasoning process in the original level-k model of BM is analogous to the procedure of the mechanism.

**Fast Deferred Acceptance**

*Round $r \geq 0$:* Each unmatched student $i$ applies to his most preferred school that has not rejected him and has not admitted enough students who all have higher priorities than $i$. Each school tentatively admits students according to its priority ranking. If all students are admitted after this round, stop the procedure.

In any round of DA an unmatched student always applies to the best school that he has not applied. But in any round of Fast DA, if the best school an unmatched student $i$ has not applied is $s$, but $s$ has admitted $q_s$ students who all have higher priorities than $i$ at $s$, then $i$ will skip $s$ and consider the school next to $s$ and so on until he finds a school
that will admit $i$ if only $i$ applies to the school in this round. So typically Fast DA runs faster than DA. But they always find the same outcome $\mu^{DA}$.

I index the first round of Fast DA by $0$ and denote the last round by $r^{FDA}$. Then for each $i$ and each $k \geq 0$ I define:

$$a^k_i \equiv \begin{cases} 
\text{the school } i \text{ applies to in round } k \text{ of Fast DA,} & \text{if } i \text{ is rejected in round } k-1, \\
\text{the school admitting } i \text{ in round } k-1 \text{ of Fast DA,} & \text{if } i \text{ is admitted in round } k-1, \\
a^r_i^{FDA}, & \text{if } k > r^{FDA}.
\end{cases}$$

That is, $a^k_i$ is the school that admits $i$ in round $k-1$, or the school $i$ applies to in round $k$. If $k$ is very high, $a^k_i$ is the school that finally admits $i$ in Fast DA. Obviously this school must be $\mu^{DA}(i)$. Now I prove that $a^k_i$ is exactly the school each $i$ reports as first choice at any $L_k$. So it is as if an $L_k$ student for any $k > 0$ runs $k$ rounds of Fast DA in his mind to find his best first choice.

**Proposition 1.** For any $P_I$, $s^k_i = a^k_i$ for all $i$ and all $k \geq 0$.

Once a student is rejected in some round of Fast DA, he must apply to a worse school in the next round. So Proposition 1 implies that each $i$ must report a weakly worse school as first choice at a higher level, but the school cannot be worse than $\mu^{DA}(i)$.

**Corollary 1.** For any $P_I$, $s^k_i \geq R_i \geq k^{k+1}$ $R_i \mu^{DA}(i)$ for all $i$ and all $k \geq 0$. For each $i$ there exists some finite $r_i \geq 0$ such that $s^k_i = \mu^{DA}(i)$ for all $k \geq r_i$.

Corollary 1 can be understood intuitively. If $i$ has a higher level, he believes the others also have higher levels. It is equivalent that he believes there is more competition in the market. So $i$ must report a weakly worse school as first choice. But since $i$ obtains $\mu^{DA}(i)$ in the most competitive situation, his first choice is never worse than $\mu^{DA}(i)$.

In the following I use $k_i$ to denote any $i$’s level and use $k_I \equiv \{k_i\}_{i \in I}$ to denote any level distribution of students. I use $\mu^{BM}_{k_I}$ to denote the outcome of BM when the level distribution is $k_I$. Recall that when $i$’s level is weakly higher than $r_i$, $i$ will report $\mu^{DA}(i)$ as first choice no matter how high his level is. So I say $i$ is **sufficiently sophisticated** if $k_i \geq r_i$. Note that the value of $r_i$ depends on $P_I$. So a sufficiently sophisticated student in one school choice problem may not be sufficiently sophisticated in another problem.
4.1 Efficiency Comparison between BM and DA

I first prove that for any $P_I$ and any $k_I$, if any $i$ is sufficiently sophisticated in $P_I$, $i$ must be admitted by $\mu^{DA}(i)$ in BM.

**Lemma 1.** For any $P_I$ and any $k_I$, if any $i \in I$ reports $\mu^{DA}(i)$ as first choice, $i$ must be admitted by $\mu^{DA}(i)$ in BM.

The proof is as follows. If some students other than $i$ also report $\mu^{DA}(i)$ as first choice in BM, then these students must also apply to $\mu^{DA}(i)$ in some round of DA. Since $i$ is admitted by $\mu^{DA}(i)$ in DA, $i$’s priority at $\mu^{DA}(i)$ is high enough. So $i$ must be admitted by $\mu^{DA}(i)$ in the first round of BM.

At least one student must be admitted in the first round of BM. By Corollary 1 and Lemma 1, the student must be admitted by a school no worse than his DA assignment. So I have the following results.

**Proposition 2.** For any $P_I$:

1. $\mu^{BM}_{k_I}$ is not strictly Pareto dominated by $\mu^{DA}$ for any $k_I$;
2. If each student is sufficiently sophisticated, $\mu^{BM}_{k_I} = \mu^{DA}$;
3. If each student is insufficiently sophisticated, $\mu^{BM}_{k_I}$ is not Pareto dominated by $\mu^{DA}$.

So $\mu^{DA}$ can Pareto dominate $\mu^{BM}_{k_I}$ for some $P_I$ and some $k_I$ only when some students are insufficiently sophisticated while the others are sufficiently sophisticated. Then in the following I prove that there must exist some insufficiently sophisticated $i$ who reports a non-truthful ranking between $\mu^{BM}_{k_I}(i)$ and $\mu^{DA}(i)$.

**Lemma 2.** For any $P_I$ and any $k_I$, if $\mu^{DA}$ Pareto dominates $\mu^{BM}_{k_I}$, there exists some insufficiently sophisticated $i$ who reports some $P'_i$ such that $\mu^{BM}_{k_I}(i) P'_i \mu^{DA}(i)$, but $\mu^{DA}(i) P'_i \mu^{BM}_{k_I}(i)$.

So if I further assume that each insufficiently sophisticated $i$ reports the true ranking between $\mu^{DA}(i)$ and any school worse than $\mu^{DA}(i)$, then $\mu^{BM}_{k_I}$ will never be Pareto dominated by $\mu^{DA}$.

**Proposition 3.** For any $P_I$ and any $k_I$, if each insufficiently sophisticated $i$ reports some $P'_i$ such that for any $s \in S$, $\mu^{DA}(i) P_i s$ implies $\mu^{DA}(i) P'_i s$, then $\mu^{BM}_{k_I}$ is not Pareto dominated by $\mu^{DA}$.
A simple manipulation strategy that satisfies the above assumption is that \( i \) reports the true preference ordering of the schools below his reported first choice. Hence \( i \) only misreports first choice. So I call it *topping strategy*.

In the example of Section 2 I show that \( \mu_{k_i}^{BM} \) Pareto dominates \( \mu^{DA} \) for some \( k_I \). Now I show that this holds for a general class of school choice problems. For any \( P_I \), I call the ordering of the schools that any \( i \) applies to in the procedure of Fast DA the *expressed preferences of \( i \) in Fast DA*. Then I have the following result.

**Proposition 4.** For any \( P_I \), if \( \mu^{DA} \) is not Pareto efficient with respect to the expressed preferences of students in Fast DA, then there exists some \( k_I \) such that \( \mu_{k_I}^{BM} \) Pareto dominates \( \mu^{DA} \).

In the proof I construct a level distribution \( k_I \) in which all students obtain their reported first choices and \( \mu_{k_I}^{BM} \) Pareto dominates \( \mu^{DA} \). For other school choice problems \( \mu_{k_I}^{BM} \) may also Pareto dominate \( \mu^{DA} \) for some \( k_I \). But since I do not assume how positive-level students report the whole preferences, I cannot answer when it must happen.

### 4.2 Comparative Statics of Sophistication Change

Since an \( Lk \) student for any \( k > 0 \) believes others are \( Lk - 1 \), he may overestimate the levels of some students but underestimate the levels of the others. As illustrated in Section 2, a student may not benefit from his sophistication in general.

Nevertheless, Lemma 1 proves that a sufficiently sophisticated \( i \) in any \( P_I \) must be admitted by \( \mu^{DA}(i) \) irrespective of the others’ levels. By contrast, the assignment of an insufficiently sophisticated \( j \) depends on the others’ levels, and it can be better or worse than \( \mu^{DA}(j) \). This may be seen as an advantage of sufficiently sophisticated students in BM if students are risk-averse.

### 5 The Informational Level-k Model of BM

In the informational level-k model an \( Lk \) student knows the level of any \( Lk' \) student if \( k' < k \), and believes the others are \( Lk - 1 \). So the strategy of an \( Lk \) student depends on

---

4Formally, if \( i \) reports \( s \) as first choice, then \( P'_I \) is the topping strategy if \( sP'_I s' \) for all \( s' \neq s \), and \( s'P'_I s'' \) if and only if \( s'P_I s'' \) for all \( s', s'' \neq s \).
the levels of the others. This is different from the original level-k model. So in this section I explicitly show such dependence in notations. In particular, for any \( P_I \) and any \( k_I \), I use \( \tilde{s}_i^k(k_I) \) to denote the first choice reported by any \( i \) at \( Lk \) for any \( 0 \leq k \leq k_i \). In the following I show that the level-k reasoning process in this model can still be understood through a variant of DA. For any \( k_I \), I define:

**Fast Deferred Acceptance**

**Round** \( r \geq 0 \): For each unmatched student \( i \), if \( k_i \geq r \), then \( i \) applies to her most preferred school that has not rejected him and not admitted enough students who all have higher priorities than \( i \). Each school tentatively admits students according to its priority ranking. If \( k_i < r \) for all unmatched \( i \), or all students are admitted after this round, stop the procedure.

The procedure of Fast DA* depends on \( k_I \), so I denote its outcome by \( \mu_{k_i}^{FDA*} \). Fast DA* is different from Fast DA in that an unmatched \( i \) cannot apply to a new school in any round \( r > k_i \). If some \( i \) is unmatched in \( \mu_{k_i}^{FDA*} \), I say \( i \) is admitted by \( \emptyset \). Let \( r_{k_i}^{FDA*} \) denote the last round of Fast DA*. For each \( i \) and each \( 0 \leq k \leq k_i \) I define:

\[
\tilde{a}_i^k(k_I) \equiv \begin{cases} 
\text{the school } i \text{ applies to in round } k \text{ of Fast DA*}, & \text{if } i \text{ is rejected in round } k - 1, \\
\text{the school admitting } i \text{ in round } k - 1 \text{ of Fast DA*}, & \text{if } i \text{ is admitted in round } k - 1, \\
\tilde{a}_{i}^{r_{k_i}^{FDA*}}, & \text{if } k > r_{k_i}^{FDA*}.
\end{cases}
\]

Then I prove that \( \tilde{s}_i^k(k_I) \) coincides with \( \tilde{a}_i^k(k_I) \).

**Proposition 5.** For any \( P_I \) and any \( k_I \), \( \tilde{s}_i^k(k_I) = \tilde{a}_i^k(k_I) \) for all \( i \) and all \( 0 \leq k \leq k_i \).

For any \( P_I \) and any \( k_I \), I denote the outcome of BM by \( \tilde{\mu}_{k_i}^{BM} \). By Proposition 5, \( \tilde{a}_i^k(k_I) \) is the last school each \( i \) applies to in Fast DA* and also the first choice reported by each \( i \) in BM. So \( \mu_{k_i}^{FDA*} \) is the matching found by the first round of BM.

**Corollary 2.** For any \( P_I \) and any \( k_I \), \( \mu_{k_i}^{FDA*} \) is the matching found by the first round of BM.

If all students are matched in \( \mu_{k_i}^{FDA*} \), \( \mu_{k_i}^{FDA*} \) must coincide with \( \mu_{k_i}^{DA} \). This happens when all students are sufficiently sophisticated. So if some \( i \) is unmatched in \( \mu_{k_i}^{FDA*} \), \( i \)
must not apply to $\mu_{DA}(i)$ in Fast DA$^*$. However, for those who are matched in $\mu_{FDA}^{*}$, they must be admitted by schools weakly better off than their DA assignments. So the following corollary holds.

**Corollary 3.** For any $P_I$ and any $k_I$, $s_i^k(k_I) R_i s_i^{k+1}(k_I) R_i \mu_{DA}(i)$ for all $i \in I$ and all $0 \leq k \leq k_i$.

### 5.1 Efficiency Comparison between BM and DA

With Corollary 3 the following results can be proved similarly as the previous section.

**Proposition 6.** For any $P_I$:

1. $\tilde{\mu}_{k_I}^{BM}$ is not strictly Pareto dominated by $\mu_{DA}$ for any $k_I$;

2. If each student is sufficiently sophisticated, $\tilde{\mu}_{k_I}^{BM} = \mu_{DA}$;

3. If each student is insufficiently sophisticated, $\tilde{\mu}_{k_I}^{BM}$ is not Pareto dominated by $\mu_{DA}$;

4. If each positive-level $i$ reports some $P'_i$ such that for any $s \in S$, $\mu_{DA}(i) P_i s$ implies $\mu_{DA}(i) P'_i s$, then $\tilde{\mu}_{k_I}^{BM}$ is not Pareto dominated by $\mu_{DA}$ for any $k_I$.

Proposition 4 in the previous section does not hold here because if all students obtain their reported first choice, $\tilde{\mu}_{k_I}^{BM}$ would coincide with $\mu_{DA}$. But as before, it can still happen that $\tilde{\mu}_{k_I}^{BM}$ Pareto dominates $\mu_{DA}$ for some $P_I$ and some $k_I$, although I cannot answer when it must happen under the best first choice assumption.

### 5.2 Comparative Statics of Sophistication Change

Now I investigate how the outcome of BM changes if I change a student’s level. I consider the situation that any student $j$’s level increases from $Lk_j$ to any $Lk'_j$ such that $k'_j > k_j$. Under the best first choice assumption I can only characterize the first choices reported by students. By Corollary 2, the outcome of Fast DA$^*$ is the matching found by the first round of BM. So in the following I will investigate how the outcome of Fast DA$^*$ changes if any $j$’s level changes. My first result is as follows.

---

5In particular, if some students are insufficiently sophisticated, a sufficiently sophisticated $i$ may report a school better than $\mu_{DA}(i)$ as first choice, but he may be rejected. So a sufficiently sophisticated student can be unmatched in $\mu_{FDA}^{*}$.
Proposition 7. For any $P_I$ and any $k_I$, if any $j \in I$ becomes $Lk_j'$ for any $k_j' > k_j$, then:

- If $\mu_{k_I}^{FDA*}(j) \neq \emptyset$, then $\hat{\mu}_{k_I}^BM = \tilde{\mu}_{(k_j',k_{j-1})}^BM$;

- If $\mu_{k_I}^{FDA*}(j) = \emptyset$, for any $i \in I$ such that $\mu_{k_I}^{FDA*}(i) \neq \emptyset$ and $\mu_{(k_j',k_{j-1})}^{FDA*}(i) \neq \emptyset$:
  - If $k_i \leq k_j + 1$, $\hat{\mu}_{k_I}^BM (i) = \tilde{\mu}_{(k_j',k_{j-1})}^BM (i)$;
  - If $k_i > k_j + 1$, $\hat{\mu}_{k_I}^BM (i) = R_i \hat{\mu}_{(k_j',k_{j-1})}^BM (i)$.

The proof is as follows. If $j$ is matched in $\mu_{k_I}^{FDA*}$, it means that $j$ obtains his reported first choice. Then becoming $Lk_j'$ does not change $j$’s strategy as well as the others’. So the outcome of BM does not change. If $j$ is unmatched in $\mu_{k_I}^{FDA*}$, then by becoming $Lk_j'$, $j$ will apply to more schools in Fast DA* than before. Then for any $i$ such that $\mu_{k_I}^{FDA*}(i) \neq \emptyset$ and $\mu_{(k_j',k_{j-1})}^{FDA*}(i) \neq \emptyset$, if $k_i \leq k_j + 1$, the level change of $j$ cannot affect the set of schools that $i$ applies to in Fast DA*. So $i$’s assignment does not change. If $k_i > k_j + 1$, since $j$ applies to more schools than before in in Fast DA*, $i$ will also apply to weakly more schools than before. So $i$’s assignment must be weakly worse off.

For any $P_I$, define $\bar{r} = \max_{k_I} r_{k_I}^{FDA*}$. So $\bar{r}$ is the biggest last round of Fast DA* for all possible level distributions. Given the set $I$ of students and the set $S$ of schools, $\bar{r} \leq |I| \cdot |S| - 1$. If any $i$’s level is weakly higher than $\bar{r}$, $i$ must be matched in the outcome of Fast DA* irrespective of the others’ levels. So I say $i$ is quasi-rational if $k_i \geq \bar{r}$. A quasi-rational student is sophisticated enough in the sense that he always obtains his reported first choice. For any $P_I$ and any $k_I$, I denote the set of quasi-rational students by $M$ and the set of the others by $N$. Then Proposition 7 implies the following corollary.

Corollary 4. For any $P_I$ and any $k_I$, if $M \neq \emptyset$ and $N \neq \emptyset$, then if any $j \in N$ becomes $Lk_j'$ for any $k_j' > k_j$,

$$\hat{\mu}_{k_I}^BM (i) = R_i \hat{\mu}_{(k_j',k_{j-1})}^BM (i)$$

for all $i \in M$.

If all students in $N$ become quasi-rational, then the outcome of Fast DA* coincides with $\mu^{DA}$. So Corollary 4 implies the following result.

Corollary 5. For any $P_I$ and any $k_I$, if $M \neq \emptyset$, all students in $M$ obtain weakly better assignments in BM than in DA.

For any $k_I$, define $\bar{k}_N = \max_{i \in N} k_i$. If $M = \emptyset$ and there exists a unique $L\bar{k}_N$ student $i$, $i$ must be matched in $\mu_{k_I}^{FDA*}$. It is because no students other than $i$ apply to schools
in round $k_N$ of Fast DA$^*$. If $i$ is matched in round $k_N - 1$, $i$ must still be matched in round $k_N$; if $i$ is unmatched in round $k_N - 1$, $i$ must apply to a school in round $k_N$ and be admitted. Then I have the following corollary. Note that if there are multiple $Lk_N$ students, the corollary may not hold.

**Corollary 6.** For any $P_i$ and any $k_i$, if $M = \emptyset$ and there is a unique $Lk_N$ student $i$, then (1) if any $j \in I \setminus \{i\}$ becomes $Lk_j'$ for any $k_j < k_i$, $\mu_{k_j}^{BM}(i) R_i \mu_{(k_j', k_{j-1})}^{BM}(i)$; (2) $i$ obtains a weakly better assignment in BM than in DA.

If $\mu_{k_i}^{FDA^*}(j) = \emptyset$, by now I have not answered whether $j$ can be better off by becoming $Lk_j'$. Through the following example I show that $j$ is not guaranteed to be better off. It is because the others in $N$ who have higher levels than $j$ may respond to $j$’s level change by using more competitive strategies.

**Example 1.** $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Each school has one seat. The preference and priority profiles are shown in Table 5.

<table>
<thead>
<tr>
<th>$P_{i_1}$</th>
<th>$P_{i_2}$</th>
<th>$P_{i_3}$</th>
<th>$P_{i_4}$</th>
<th>$P_{i_5}$</th>
<th>$P_{i_6}$</th>
<th>$\pi_{s_1}$</th>
<th>$\pi_{s_2}$</th>
<th>$\pi_{s_3}$</th>
<th>$\pi_{s_4}$</th>
<th>$\pi_{s_5}$</th>
<th>$\pi_{s_6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$s_2$</td>
<td>$s_3$</td>
<td>$s_4$</td>
<td>$s_1$</td>
<td>$s_1$</td>
<td>$i_6$</td>
<td>$i_3$</td>
<td>$i_4$</td>
<td>$i_5$</td>
<td>$i_2$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>$s_3$</td>
<td>$s_4$</td>
<td>$\vdots$</td>
<td>$i_1$</td>
<td>$i_3$</td>
<td>$i_4$</td>
<td>$i_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_5$</td>
<td>$s_3$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$i_2$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_6$</td>
<td>$s_5$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5

If $i_1$ is $L0$, $i_2$ is $L2$, and all others are quasi-rational, the first choices reported by all students at all levels are shown in Table 6. If $i_1$ becomes quasi-rational, the first choices are shown in Table 7.

<table>
<thead>
<tr>
<th>Level 0:</th>
<th>$i_1 \rightarrow s_1$</th>
<th>$i_2 \rightarrow s_2$</th>
<th>$i_3 \rightarrow s_3$</th>
<th>$i_4 \rightarrow s_4$</th>
<th>$i_5 \rightarrow s_1$</th>
<th>$i_6 \rightarrow s_1$;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1:</td>
<td>$i_2 \rightarrow s_2$</td>
<td>$i_3 \rightarrow s_3$</td>
<td>$i_4 \rightarrow s_4$</td>
<td>$i_5 \rightarrow s_4$</td>
<td>$i_6 \rightarrow s_1$</td>
<td>;</td>
</tr>
<tr>
<td>Level 2:</td>
<td>$i_2 \rightarrow s_2$</td>
<td>$i_3 \rightarrow s_3$</td>
<td>$i_4 \rightarrow s_3$</td>
<td>$i_5 \rightarrow s_4$</td>
<td>$i_6 \rightarrow s_1$</td>
<td>;</td>
</tr>
<tr>
<td>Level $k \geq 3$:</td>
<td>$i_3 \rightarrow s_2$</td>
<td>$i_4 \rightarrow s_3$</td>
<td>$i_5 \rightarrow s_4$</td>
<td>$i_6 \rightarrow s_1$</td>
<td>;</td>
<td></td>
</tr>
</tbody>
</table>

Table 6
Suppose all students at positive levels use topping strategies, then the outcomes of BM are shown below. It is easy to see that \( i_1 \) is worse off by becoming quasi-rational.

<table>
<thead>
<tr>
<th>Level 0:</th>
<th>( i_1 \rightarrow s_1 )</th>
<th>( i_2 \rightarrow s_2 )</th>
<th>( i_3 \rightarrow s_3 )</th>
<th>( i_4 \rightarrow s_4 )</th>
<th>( i_5 \rightarrow s_1 )</th>
<th>( i_6 \rightarrow s_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1:</td>
<td>( i_1 \rightarrow s_2 )</td>
<td>( i_2 \rightarrow s_2 )</td>
<td>( i_3 \rightarrow s_3 )</td>
<td>( i_4 \rightarrow s_4 )</td>
<td>( i_5 \rightarrow s_4 )</td>
<td>( i_6 \rightarrow s_1 )</td>
</tr>
<tr>
<td>Level 2:</td>
<td>( i_1 \rightarrow s_2 )</td>
<td>( i_2 \rightarrow s_5 )</td>
<td>( i_3 \rightarrow s_3 )</td>
<td>( i_4 \rightarrow s_3 )</td>
<td>( i_5 \rightarrow s_4 )</td>
<td>( i_6 \rightarrow s_1 )</td>
</tr>
<tr>
<td>Level 3:</td>
<td>( i_1 \rightarrow s_2 )</td>
<td>( i_3 \rightarrow s_2 )</td>
<td>( i_4 \rightarrow s_3 )</td>
<td>( i_5 \rightarrow s_3 )</td>
<td>( i_6 \rightarrow s_1 )</td>
<td></td>
</tr>
<tr>
<td>Level ( k \geq 4 ):</td>
<td>( i_1 \rightarrow s_6 )</td>
<td>( i_3 \rightarrow s_2 )</td>
<td>( i_4 \rightarrow s_3 )</td>
<td>( i_5 \rightarrow s_4 )</td>
<td>( i_6 \rightarrow s_1 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 7

But if \( j \) has the highest level among \( N \), then \( j \) must be weakly better off if he uses a strategy that satisfies a mild condition at \( L_k j \). Formally, I use \( P^k j \) and \( P'^k j \) to denote the preferences reported by \( j \) at \( L_k j \) and \( L_k' j \) respectively. Then if \( s \) is the first choice in \( P'^k j \), I say \( P'^k j \) satisfies worse-rank invariance if for any \( s' \) such that \( s P^j s' \), \( s' \) is ranked at the same position in \( P^k j \) and \( P'^k j \).

**Proposition 8.** For any \( P_I \) and any \( k_I \), if any \( j \in N \) such that \( k_j = \bar{k}_N \) becomes \( L_k j \) for any \( k_j' > k_j \) and his reported preferences satisfies worse-rank invariance, then

\[
\tilde{\mu}_{BM}^{k} \left(P^{k,j}_{L_{k_j}'}(j) \right) R_j \tilde{\mu}_{BM}^{k} (j) \]

If \( j \) uses topping strategies at all positive levels, the worse-rank invariance condition is always satisfied. If \( j \) becomes quasi-rational, since \( j \) must be admitted by his reported first choice, \( j \) must be weakly better off even if worse-rank invariance is violated.

### 6 Discussion

#### 6.1 Implications of the Two Level-k Models

In both level-k models of BM I show that the iterated reasoning process is analogous to the procedure of a variant of DA. Based on this I show that in general BM is not (strictly)
Pareto dominated by DA. Since the two models make extreme assumptions on the beliefs
of positive-level students, I believe similar results will hold in any other level-k model of
BM that lies in between the two extreme ones.

Ergin and Sönmez (2006) assume students are rational and prove that every NE
outcome of BM is a stable matching with respect to the true preferences of students.
Since the outcome of DA is the best stable matching $\mu^{DA}$ for students, it implies that BM
cannot be more efficient than DA, and is Pareto dominated by DA if any NE outcome
other than $\mu^{DA}$ is realized. In the two level-k models if I let students be sufficiently
sophisticated to approximate rationality, then the outcome of BM will coincide with
$\mu^{DA}$. This implies that if students are more likely to use the level-k reasoning than the
circular equilibrium reasoning, then students are more likely to coordinate on $\mu^{DA}$ as long
as they have sufficient sophistication.

The two level-k models are different in the advantage of highly sophisticated students.
The difference implies that a student benefits from his sophistication in BM only if he
has a correct belief about others’ levels and his level is high. It is interesting to compare
my results with that of Pathak and Sönmez (2008). They assume a simple environment
in which some students are naive while the others are rational. By assuming the best
NE outcome of BM is always realized, they prove that rational students benefit from
rationality and there exists a conflict of interest between naive students and rational
students. This dichotomous distribution can be seen as a special case of my models if
students are either $L0$ and have high levels. Indeed, I prove that if students are either $L0$
or quasi-rational, then the outcome of BM in the informational level-k model is exactly
the best NE outcome of BM if quasi-rational students are treated as rational.

**Proposition 9.** For any $P_I$ and any $k_I$, if $N, M \neq \emptyset$ and $k_N = 0$, then $\tilde{\mu}^{BM}$ is the best
NE outcome of BM when $N$ are naive and $M$ are rational.

In Appendix B I use a new method to characterize the set of NE outcomes of BM
when $N$ are naive and $M$ are rational in a simple way. Proposition 9 is a corollary of the
result. Then by Proposition 7, if any $j \in N$ becomes quasi-rational, all students in $M$ are
weakly worse off. Since $k_N = 0$, each $j \in N$ has the highest level in $N$. By Proposition
8 if any $j \in N$ becomes quasi-rational, $j$ must be weakly better off. Hence I obtain
the result of Pathak and Sönmez as a corollary.\(^6\) The level-k models provide a richer

\(^6\)In Appendix B I show that the other results of Pathak and Sönmez can be proved easily by my
setting than the dichotomous distribution to study the relation between sophistication and welfare in BM, and by varying the knowledge of sophisticated students I clarify the conditions for the result.

6.2 Remarks on My Assumptions

$L_0$ strategy  I assume that $L_0$ students report true preferences. It is different from many other papers that assume $L_0$ players play a random strategy. The difference is caused by different features of the games that level-$k$ models are applied to. Intuitively, $L_0$ strategy captures the instinct response of a player to a game. When it is unclear what strategy is a reasonable instinct response in a game, the literature often assumes random strategy. For example, in the “$p$-beauty contest” game each player is asked to propose an integer between 0 and 100. The winner is the one whose proposal is closest to a multiple $p$ of the group average. In this game it is unclear which integer is more likely to be chosen by a $L_0$ player, so we often assume that $L_0$ players uniformly choose an integer.

But in some games it is believed that some strategies are more likely to become instinct responses than the others. These strategies are called salient strategies. For example, Crawford and Iriberri (2007a) point out the framing effects in the experiments of “hide-and-seek” games. By suitably adapting $L_0$ behavior to salient strategies, they show that the level-$k$ model can well explain the experimental dataset. Arad and Rubinstein (2012) conduct experiments of the “11-20” game to estimate the levels of players. In the game each of two players reports an integer between 11 and 20 and obtains an amount of dollars equaling his report; a player can win additional 20 dollars if his report is one less than the other’s. Since the game rule is straightforward, the authors argue that it is very natural for a naive player to report 20, and any report below 20 implies that some strategic reasoning is made.

I believe BM is closer to the 11-20 game than to the $p$-beauty contest game. First, when students participate in school choice, they are asked to report preferences. So a naive student without any reasoning will naturally report his true preferences. Second, by participating in school choice students expect to be admitted by schools, and this is guaranteed in many cities. So BM is more like the 11-20 game in which every player wins a prize than the $p$-beauty contest game in which generically only one player wins a prize.
Then a naive student is not likely to think about the linkage between his report and his assignment in BM. So I believe my assumption on $L0$ strategy is reasonable.

$Lk$ strategy I assume that an $Lk$ student for any $k > 0$ manipulates BM through misreporting first choice. I believe this is a reasonable assumption for the following reasons. First, first choice plays the most role in determining a student’s assignment in BM, so it is natural for him to focus on first choice. Second, in practice students may be advertised to focus on first choice. For example, the reference material of Boston provided to students in 2004 suggests students to strategically choose first choice; in Seattle and Tampa-St. Petersburg such suggestions appear in local press (Abdulkadiroğlu et al., 2005). Third, this assumption is consistent with experimental evidence. Chen and Sönmez (2006) conduct experiments and find that in BM 70.8% of students receive their reported first choices, but only 28.5% receive their true first choices. This implies that a significant proportion of manipulation is realized by misreporting first choice. Lastly, in the level-k reasoning process an $Lk$ student for any $k > 1$ is uncertain about the whole preferences reported by the others at $Lk-1$. If he considers the worst case, he will assume the others optimally choose their first choices, then optimally choose his first choice.

7 Simulation

In previous sections I consider all possible levels of students. When I define sufficient sophistication and quasi-rationality, the threshold levels can be high. But many experiments of level-k models have found that subjects’ levels are often between $L0$ and $L3$. So in this section I take this fact into account and do simulations to compare BM and DA.

Setup There are $x$ students and $y$ schools. Each school has $q$ seats and $q \times y = x$. The utility function $U_i$ of each student $i$ and the utility function $U_s$ of each school $s$ are generated as follows:

$$U_i(s) = \alpha U(s) + (1 - \alpha) \epsilon_i(s),$$

$$U_s(i) = \beta U(i) + (1 - \beta) \epsilon_s(i).$$

Here $U(s)$ and $U(i)$ are the common values of each $s$ and each $i$ respectively. $\epsilon_i(s)$ is the

---

7Although I call $U_s$ the utility function of school $s$, schools actually do not have preferences over students. I define $U_s$ to generate the priority ranking of $s$. 

23
private value of each $s$ in the eyes of each $i$ and $\epsilon_s(i)$ is the private value of each $i$ in the eyes of each $s$. All $U$ and $\epsilon$ are independently and identically drawn from the uniform distribution on $[0, 1]$. $\alpha \in [0, 1]$ is the correlation coefficient in the utilities of students and $\beta \in [0, 1]$ is the correlation coefficient in the utilities of schools. When all agents’ utility functions are generated, the preferences of students and priorities of schools are generated as follows:

$$P_i: s_a P_i s_b \iff U_i(s_a) > U_i(s_b),$$

$$\pi_s: i_a \pi_s i_b \iff U_s(i_a) > U_s(i_b).$$

**Parameter** I consider two market sizes: in small markets $x = 20$, $y = 4$ and $q = 5$; in large markets $x = 1000$, $y = 20$ and $q = 50$. Lab experiments often involve only a small number of participants, while practical markets often involve a large number of participants. A couple of papers show that large markets can perform differently than small ones (Azevedo and Leshno, 2015; Lee, 2016; Che and Tercieux, 2015). So I use two market sizes to accommodate both environments and also check the robustness of the simulation results.

For each market size I vary $\alpha$ and $\beta$ from 0 to 1 in steps of .2. Hence $\alpha, \beta \in \{0, .2, .4, .6, .8, 1\}$. For each market size and each vector of $(\alpha, \beta)$ I randomly generate 1000 markets as well as the preferences and priorities in these markets. To generate the levels of students I choose the Poisson distribution with a mean of 2.\(^8\) Under this distribution the probabilities for $L_0$ to $L_4$ are .135, .271, .271, .180, .090, respectively. So most students’ levels are between $L_0$ to $L_3$. In previous sections I do not assume how positive-level students report the whole preferences. In simulations I consider two settings: in the first setting positive-level students use topping strategies; in the second setting positive-level students uniformly and randomly report the orderings of the schools below first choice, which I call random strategy. Then I run the two level-k models of BM and DA, and compare their outcomes.

**Result** For any parameter vector, I report the following indexes to compare the outcomes of BM and DA. I first report the percentage of students in the 1000 markets who

---

\(^8\)Camerer, Ho and Chong (2004) use the Poisson Cognitive Hierarchy model to estimate multiple games and find the median estimation of the Poisson mean is 1.61. Arad and Rubinstein (2012) find the best estimate of the Poisson mean for the “11-20” game is 2.36.
are better off in BM and the percentage of students who are worse off in BM. To measure the average welfare of students in a mechanism I calculate the average rank of students’ assignments in their true preferences. I then report the difference between the average rank in DA and the average rank in BM. If the difference is positive, students are on average better off in BM; vice versa. I also calculate the variance of the ranks of students’ assignments in both mechanisms, and report the difference between the variance in BM and the variance in DA. If the difference is positive, it means the assignments in BM are more diverse than in DA. Last, I report the opposite of the correlation coefficient between the ranks of students’ assignments in BM and their levels. If the value is positive, it means that students’ welfare is positively correlated with their levels in BM. In DA the correlation coefficient is expected to be zero, so I do not report it.

Figure 1: Simulation results in small markets

---

9When a student obtains a better assignment, the rank of his assignment in his preferences is smaller. In particular, if he obtains his most preferred school, the rank is 1.
Figure 1 and Figure 2 are the simulations results for the original level-k model of BM in small and large markets respectively. In each figure the left two columns correspond to the setting that positive-level students use topping strategies, while the right two columns correspond to the setting that positive-level students use random strategies. Each subfigure corresponds to one reported index. The horizontal axis is the value of $\alpha$, the vertical axis is the value of the reported index, and the six lines correspond to the six values of $\beta$. The simulation results for the informational level-k model have no qualitative differences, so I present them in Appendix C. Ashlagi, Kanoria and Leshno (2015) show that unbalanced markets can perform very differently from balanced markets. To examine the robustness of my simulation results I consider unbalanced markets by setting $U(s) = \epsilon_i(s) = 0$ for all $i$ and some $s$. In this way $s$ becomes the worst school in all students’ preferences and plays the role of “unmatched”. I present the simulation results in Appendix C. Unbalanced markets make some results sharper, but the qualitative conclusions do not change.
The observations from Figure 1 and Figure 2 are summarized as follows. First, the outcome of BM is often different from that of DA, but neither clearly dominates the other. In both small and large markets the percentage of students who are better off in BM than in DA and the percentage of students who are worse off in BM than in DA are both significantly above 0. In large markets the two percentages are often above 10% and can reach 40%. Second, it seems that BM is slightly better than DA. Specifically, the percentage of better off students in BM is often higher than the percentage of worse off students in BM, which implies that more students prefer BM to DA. The average rank difference is also often positive although it is very small.\textsuperscript{10} This means that the welfare of an average student is slightly higher in BM. But the variance difference implies that the outcome of BM is often more diverse than that of DA, but the conclusion is not robust to the strategies of positive-level students and market sizes. Last, both figures (and other figures in Appendix C) show that it is robust that welfare and level are positively correlated in BM. In the original level-k model, the correlation coefficient is small in small markets, but ranges from .1 to .35 for most parameter values in large markets. In the informational level-k model shown in Appendix C, the correlation coefficient is significantly above zero in small markets, and ranges from .1 to .4 in large markets. This result suggests that students benefit from their sophistication statistically although their levels are low and beliefs about others’ levels are incorrect.

As discussed in Introduction, my simulation results are consistent with some empirical studies (He, 2014; Calsamiglia, Fu and Güell, 2015) that estimate that replacing BM with DA in some cities will make more students worse off than making students better off, and will reduce the welfare of an average student and most sophisticated students. So I conjecture that a suitable adaption of the level-k model can explain the strategies of students in empirical datasets. This is an interesting direction for future research.

\textsuperscript{10}In Figure 2 if positive-level students use topping strategies, the average rank difference can be negative. But in unbalanced markets of Appendix C the difference is always nonnegative even though positive-level students use topping strategies. So I conclude that the average rank difference is often positive.
8 Extension

8.1 Constrained School Choice

In practice there are many schools in a school system, but students are often constrained to list only a few schools in their submitted preferences. Many examples can be found in Haeringer and Klijn (2009) and Pathak and Sönmez (2013). In this environment it is impossible for students to report true preferences in any school choice mechanism.\textsuperscript{11} So in the following I discuss how students may report preferences if they use the level-k reasoning in BM and DA.

I first argue that my previous characterizations of the two level-k models of BM still hold in constrained school choice. This is because in BM students essentially compete on their first choices. By contrast, in DA students compete on their whole preferences. So when an $Lk$ student infers the best strategy of an $Lk - 1$ student, he is not sure which strategy the $Lk - 1$ student may use. Hence a level-k model of DA will highly depend on the selection assumption on students’ best strategies at all levels, and it is not clear which assumption is reasonable. To provide an example in Appendix D I analyze an original level-k model of DA by assuming $L0$ students report true preferences up to the constraint and positive-level students use topping strategies. The level-k reasoning process in this model is very similar to the one in BM, and simulations show that DA in this model perform better than BM. However, by Haeringer and Klijn (2009) topping strategies are dominated strategies for positive-level students. So if the topping strategy assumption is believed to be unreasonable, then Appendix D should not be taken for granted.

8.2 Coarse Priority and Incomplete Information

Previous sections assume that school priorities are strict and students have complete information of preferences and priorities. But in practice schools often prioritize students coarsely and students often have incomplete information of the others’ preferences. So in the following I discuss how students may do the level-k reasoning in BM when the previous assumptions are relaxed.

To simplify the analysis I assume each school prioritizes all students randomly ac-

\textsuperscript{11}Of course if the number of schools a student has interest in is smaller than the constraint, the student is not constrained. Here I consider the case that students are really constrained by the restriction.
According to a probability distribution type of each student \(i\) by \(v^i \equiv (v^i_s)_{s \in S}\) where \(v^i_s\) is the utility of obtaining \(s\). \(v^i\) is also called the type of \(i\). \(v^i\) is drawn from the type space \(\mathcal{V} \equiv \{(v^i_s)_{s \in S} \in [0,1]^{|S|} : v^i_s \neq v^j_s, \forall s,s' \in S\}\) according to a probability distribution \(f\). I assume \(f\) is public information and has full support. That is, \(f(v^i) > 0\) for all \(v^i \in \mathcal{V}\). Let \(P_v\) be the preference ordering induced by any \(v \in \mathcal{V}\).

I only discuss the original level-k model of BM. I use \(P^k_v\) to denote the preferences reported by any type-\(v\) students at any \(Lk\). As before \(L0\) students report true preferences. So \(P^0_v = P_v\). I assume positive-level students are risk-neutral, so they choose strategies to maximize their expected utilities. Then for any \(v \in \mathcal{V}\) and any \(k > 0\),

\[
P^k_v \equiv \arg \max_{P \in \mathcal{P}} EU^k_v(P^*),
\]

where \(EU^k_v(P^*)\) is the expected utility of type-\(v\) students by reporting \(P^*\). Specifically, let \(\mu^{BM}(P^k_{v-1}, P^*)\) be the random outcome of BM if a type-\(v\) students \(i\) reports \(P^*\) and the others report \(P^k_{v-1}\). Let \(\mu^{BM}(P^k_{v-1}, P^*)(i)(s)\) be the probability that \(i\) obtains any \(s \in S\). Then,

\[
EU^k_v(P^*) = \int_{v^{-i} \in \mathcal{V}^{[l]-1}} \sum_{s \in S} [\mu^{BM}(P^k_{v-1}, P^*)(i)(s) \cdot v_s] f(v^{-i}) dv^{-i}
= \sum_{s \in S} \left[ \int_{v^{-i} \in \mathcal{V}^{[l]-1}} \mu^{BM}(P^k_{v-1}, P^*)(i)(s) f(v^{-i}) dv^{-i} \right] v_s
\]

When \(k = 1\), \(P^0_{v-i}\) are the true preferences of the students other than \(i\). So \(EU^1_v(P^*)\) is well-defined. Since \(f(v^{-i}) = \prod_{j \neq i} f(v^j) > 0\) for all \(v^{-i} \in \mathcal{V}^{[l]-1}\), with probability one there is a unique \(P^1_v\) that maximizes \(EU^1_v(P^*)\). If \(P^1_v\) is not unique, choose an arbitrary best strategy. When \(k \geq 2\), since \(f(v^{-i}) > 0\) for all \(v^{-i} \in \mathcal{V}^{[l]-1}\) and \(P^{k-1}_v\) is generally unique, \(EU^k_v(P^*)\) is still well-defined. Then \(P^k_v\) is still generically unique. If it is not unique, choose an arbitrary best strategy.

The strategies of students in this model depend on students’ beliefs and cardinal utilities. Without more assumptions it is hard to characterize more properties of their strategies and compare the outcome of BM with that of DA.

**Common Ordinal Preferences**  Suppose there are \(m\) schools \(s_1, \ldots, s_m\) and all students prefer \(s_i\) to \(s_{i+1}\). That is, \(v^i_{s_1} > v^i_{s_2} > \cdots > v^i_{s_m} > 0\) for all \(i\). Then by believing
the others report $s_1$ as first choice, an $L1$ student $i$ who does not very prefer $s_1$ to $s_2$ will report $s_2$ as first choice. That is, if $v_{i1}^{s_1} \geq v_{i2}^{s_2}$, $i$ reports $s_1$ as first choice; otherwise $i$ reports $s_2$ as first choice. For any $L2$ student, if he reports $s_1$ as first choice at $L1$, he must still report $s_1$ as first choice. But if he reports $s_2$ as first choice at $L1$, he may change to report $s_1$ or $s_3$ as first choice, depending on his utility intensities. If I repeat this process for higher levels, it is as if students coordinate their strategies in the level-k reasoning process. Abdulkadiroğlu, Che and Yasuda (2011) analyze this special environment and prove that students will coordinate their strategies well in any symmetric Bayesian NE of BM. Here I show that if students have bounded rationality and use the level-k reasoning, they can still coordinate their strategies to some extent. In practice if students’ preferences are correlated, it is more likely that they have common preferences over a few top schools and diverse preferences over the remaining schools. Then students do not need to have high levels to coordinate their strategies on top schools.\footnote{Abdulkadiroğlu, Che and Yasuda (2011) also consider the complete information environment and assume some students are naive while the others are rational. In their on-line appendix they prove that if any naive student becomes rational, the other naive students will be weakly worse off in the unique NE outcome of BM. So there is a conflict of interest among naive students. In Appendix E I show that this result is incorrect.}

9 Related Literature

There are a lot of papers about school choice and matching theory. In this section I discuss a few closely related papers. Abdulkadiroğlu, Che and Yasuda (2011) consider the incomplete information and no priorities environment in which students have identical ordinal preferences, and prove that in any symmetric Bayesian NE of BM students are better off than in DA. The driving force behind their result is the stylized priority and preference assumption, not incomplete information. Specifically, in the no priorities environment a student does not care about the identities of the others as long as he knows the cardinal utility distribution in the student population. So assuming common knowledge of the cardinal utility distribution has the same effect of assuming complete information. Troyan (2012) generalizes the above result by relaxing the no priorities assumption to

\footnote{For example, to reach coordination in the motivating example of Abdulkadiroğlu, Che and Yasuda (2011) that contains three students and three schools, $i_1, i_2$ can have any levels and $i_3$ can have any positive level.}
coarse priorities and using some ex ante efficiency criterion. Featherstone and Niederle (2014) use experiments to test the above result. They design a simple environment in which there is a unique non-truth-telling Bayesian NE in BM. But they find that subjects fail to coordinate on the unique equilibrium even with feedback and repetition. So the benefit of BM suggested by Abdulkadiroğlu, Che and Yasuda (2011) may be hard to realize in practice.

To test whether sophisticated students have an advantage in BM, Basteck and Mantovani (2016) do experiments of BM in lab and measure the cognitive ability of subjects by tests. Although the test scores of subjects have a wide range, they classify subjects only into two groups: those with top half scores are high ability and the remaining are low ability. By matching subjects’ assignments in BM with their ability group, they find that low-ability subjects earn significantly lower payoffs than high-ability ones in BM, but the difference is small in DA. Meanwhile, the average payoff of subjects in BM is higher than that in DA. It is interesting that my simulation results have similar features. So a level-k model may be able to explain the dataset in this study.

Some scholars collect practical datasets from cities to estimate students’ strategies or types in BM. Using the dataset from Beijing of China, He (2014) estimates a structural model that accounts for heterogeneous sophistication. He rejects the hypothesis that students are all naive or all highly sophisticated. By doing counter-factual analyses of replacing BM with DA he estimates that sophisticated students will be worse off, but the effect on naive students is ambiguous, depending on the sophistication distribution of students. Using the dataset from Barcelona of Spain, Calsamiglia, Fu and Güell (2015) estimate both students’ preferences and their sophistication types. They estimate that replacing BM with DA would make fewer than 10% of students better off but make 28% of students worse off. The welfare loss of an average student is equivalent to 60 euro. Dur, Hammond and Morrill (2015) obtain an interesting dataset from Wake County, North Carolina of the US. In that dataset students have two weeks to submit preferences to a website. Once submitted, students can also revise reports as many times as they want before the deadline. It is interesting that upon each visit of the website a student can see a summary of the number of students who has reported each school as first choice. So if a student visits the website more often, he will know more about the others’ reports. In the dataset the average visiting times is 4.61 and the standard deviation is 8.65.
of students visit the website more than once. Dur et al. treat those visiting once as naive and treat the remaining as sophisticated with differentiating their heterogeneity. They find that sophisticated students do have advantages over naive ones in the dataset.

Haeringer and Klijn (2009) analyze BM and DA in constrained school choice. They prove that the set of NE outcomes in BM is equal to the set of stable matchings, but the set of NE outcomes in DA is a superset of stable matchings. So it is hard to compare BM and DA based on their NE outcomes. Calsamiglia, Haeringer and Klijn (2010) use experiments to study the impact of constraints. They find that constraints significantly reduce the efficiency of both BM and DA and the proportion of truth telling in DA. But DA is more efficient and stable than BM.

Top Trading Cycle (TTC, Shapley and Scarf, 1974) is a popular matching mechanism, but is not widely used in school choice. One reason is that it is hard to explain the role of priorities in TTC to schools and students. So I do not analyze it in the paper. TTC is strategy-proof, so even though students have heterogeneous levels and follow the level-k reasoning, they still report true preferences in TTC.

10 Conclusion

In this paper I study the behavior of heterogeneously sophisticated students in school choice. In particular, I use two level-k models to study the strategies of students in BM, and show that the level-k reasoning process in BM is analogous to the procedure of DA. Then I prove that BM is no less efficient than DA, and sophisticated students have an definite advantage in BM only when their levels are high enough and they know others’ levels well. I do simulations to further quantify the difference between BM and DA. Simulation results show that BM performs slightly better than DA, but neither of them clearly dominates the other. Simulation results also show that students statistically benefit from their sophistication even though their levels are low and beliefs about others’ levels are incorrect. Simulation results are consistent with some recent empirical estimations, so I conjecture that level-k models potentially explain these empirical datasets. This is left for future research.

My purpose in this paper is not to oppose the replacement of BM with DA that happened in many cities. I want to provide models to understand the effect of heterogeneous
sophistication in school choice and to evaluate popular arguments that support this replacement. My findings imply that the straightforward incentive in DA is the most valid argument to support its replacement of BM. Indeed, Pathak and Sönmez (2013) present multiple school choice reforms in which the main motivation is to reduce manipulation. Pathak (2016) emphasizes this point in his keynote talk in the 11th Econometric Society World Congress. However, it should not be ignored that some empirical and experimental studies have found that some players attempt to manipulate DA. For example, in the experiments of Chen and Sönmez (2006), 36% of subjects attempt to manipulate DA. In the survey conducted by Rees-Jones (2015) from the participants in the 2012 National Resident Matching Program, around 5% of respondents report that they attempt to manipulate DA. So it seems that some players do not understand the dominant strategy property of DA. Li (2015) proposes a refined notion of strategy-proofness called obvious strategy-proofness. In an obviously strategy-proof mechanism players are more likely to understand that truth-telling is a dominant strategy than in a non-obviously strategy-proof mechanism. However, Ashlagi and Gonczarowski (2015) prove that no stable mechanism is obviously strategy-proof. This raises new questions: how do different sophistication levels affect students’ understanding of the dominant strategy property of DA, and when they do not understand, what strategies will they use? The answers are left for future research.

References


Ashlagi, Itai, and Yannai A Gonczarowski. 2015. “No stable matching mechanism is obviously strategy-proof.” *working paper*.


Che, Yeon-Koo, and Olivier Tercieux. 2015. “Payoff equivalence of efficient mechanisms in large matching markets.” *working paper*.


He, Yinghua. 2014. “Gaming the boston school choice mechanism in beijing.” *Manuscript, Toulouse School of Economics*.


Rees-Jones, Alex. 2015. “Suboptimal behavior in strategy-proof mechanisms: Evidence from the residency match.” *working paper*.


A Omitted Proofs

Proof of Proposition 1

For any \( k \geq 0 \), if each \( i \) reports \( s_i^k \) as first choice, denote the matching found by the first round of BM by \( \mu^k \). I prove by induction that \( \mu^k \) is just the matching found by round \( k \) of Fast DA. Then the proposition follows.

- At \( L_0 \), \( s^0_i \) is the most preferred school of \( i \). In round 0 of Fast DA \( i \) applies to his most preferred school. So it is obvious that \( s^0_i = a^0_i \) and \( \mu^0 \) is the matching found by round 0 of Fast DA.

- Assume that for all \( r \leq k \) for some \( k \geq 0 \) it is true that \( \mu^r \) is the matching found by round \( r \) of Fast DA. Now I consider \( k + 1 \).

If \( \mu^k(i) = s_i^k \), which means \( i \) is admitted by his reported first choice at \( Lk \) in \( \mu^k \), then \( s_i^k \) must still be \( i \)'s best obtainable school at \( Lk + 1 \). \( \mu^k(i) = s_i^k \) implies that \( i \) is matched in round \( k \) of Fast DA. So \( s_i^{k+1} = a_i^{k+1} \). If \( \mu^k(i) = \emptyset \), \( i \) will report a school worse than \( s_i^k \) at \( Lk + 1 \). By the induction assumption the school is also the one \( i \) will apply to in round \( k + 1 \) of Fast DA. So \( s_i^{k+1} = a_i^{k+1} \).

Then \( \mu^{k+1} \) must coincide with the matching found by round \( k + 1 \) of Fast DA.

- By induction \( s_i^k = a_i^k \) for all \( i \) and all \( k \leq r_{FDA} \). For all \( k > r_{FDA} \), \( i \) will always report \( s_i^{r_{FDA}} \) as first choice at \( Lk \). So \( s_i^k = a_i^k \) for all \( i \) and all \( k > r_{FDA} \).

Proof of Lemma 2

I prove it by contradiction. Suppose every insufficiently sophisticated \( i \) such that \( \mu^{DA}(i) P_i \mu^{BM}_{k_i}(i) \) reports a preference ordering \( P'_i \) such that \( \mu^{DA}(i) P'_i \mu^{BM}_{k_i}(i) \). Let \( i_1 \) be an arbitrary such student. Since \( \mu^{DA}(i_1) P'_i \mu^{BM}_{k_i}(i_1) \), \( i_1 \) must be rejected by \( \mu^{DA}(i_1) \) in some round of BM. Denote the round by \( r^1 \). Then \( \mu^{DA}(i_1) \) must admit \( q_{\mu^{DA}(i_1)} \) students in \( \mu^{BM}_{k_i} \), and there must exist some \( i_2 \) admitted by \( \mu^{DA}(i_1) \) in \( \mu^{BM}_{k_i} \) but \( \mu^{DA}(i_2) \neq \mu^{DA}(i_1) \). Since \( \mu^{BM}_{k_i} \) Pareto dominates \( \mu^{DA} \), \( i_2 \) must prefer \( \mu^{DA}(i_2) \) to \( \mu^{DA}(i_1) \). Then by assumption \( \mu^{DA}(i_2) P_{i_2}' \mu^{DA}(i_1) \). Since \( i_2 \) must apply to \( \mu^{DA}(i_1) \) in a round no latter than \( r^1 \), \( i_2 \) must apply to \( \mu^{DA}(i_2) \) and be rejected in some earlier round \( r^2 \) such that \( r^2 < r^1 \). By the same argument as before, there must exist some \( i_3 \) who is admitted by \( \mu^{DA}(i_2) \) in \( \mu^{BM}_{k_i} \) but \( \mu^{DA}(i_3) \neq \mu^{DA}(i_2) \). Then \( i_3 \) must prefer \( \mu^{DA}(i_3) \) to \( \mu^{DA}(i_2) \) and is rejected by \( \mu^{DA}(i_3) \).
in BM. Denote by $r^x$ the earliest round in which some student $i_x$ is rejected by $\mu^{DA}(i_x)$ in BM. Then there must exist some student $i_{x+1}$ who is admitted by $\mu^{DA}(i_{x+1})$ but $\mu^{DA}(i_{x+1}) \neq \mu^{DA}(i_x)$. As before, $i_{x+1}$ must apply to $\mu^{DA}(i_{x+1})$ and be rejected in a round earlier than $r^x$. But this contradicts the assumption that $r^x$ is the earliest round in which some $i_x$ is rejected by $\mu^{DA}(i_x)$.

**Proof of Proposition 4**

Let $\tilde{P}_i$ be the expressed preferences of $i$ in Fast DA. Let $\tilde{\mu}$ be a Pareto efficient matching that Pareto dominates $\mu^{DA}$ with respect to $\{\tilde{P}_i\}_{i \in I}$. Denote by $\tilde{I} \equiv \{i \in I : \tilde{\mu}(i) \tilde{P}_i \mu^{DA}(i)\}$ the set of students who are better off in $\tilde{\mu}$ with respect to $\{\tilde{P}_i\}_{i \in I}$ than in $\mu^{DA}$. Since every $i \in \tilde{I}$ must apply to $\tilde{\mu}(i)$ in some round of Fast DA, $i$ must report $\tilde{\mu}(i)$ as first choice at some level. Then for the level distribution in which every $i \in \tilde{I}$ is at the level of reporting $\tilde{\mu}(i)$ as first choice and every $j \in I \setminus \tilde{I}$ is at the level of reporting $\mu^{DA}(j)$ as first choice, the outcome of BM is just $\tilde{\mu}$. Since $\tilde{\mu}$ must Pareto dominate $\mu^{DA}$ with respect to $P_I$, the proof is finished.

**Proof of Proposition 5**

The proof is similar to that of Proposition 1. The only difference is that every $Lk$ student can see the level of every $Lk'$ student if $k' < k$. Hence in Fast DA* every $Lk'$ student cannot apply to new schools after round $k'$, and this fact is known to every $Lk$ student.

**Proof of Proposition 6**

1. By Corollary 3, a student $i$ at any level must report a school no worse than $\mu^{DA}(i)$ as first choice. So as before $\tilde{\mu}_{k_i}^{BM}$ must not be strictly Pareto dominated by $\mu^{DA}$.

2. If each student is sufficiently sophisticated, the outcome of Fast DA* must coincide with $\mu^{DA}$. So $\tilde{\mu}_{k_i}^{BM} = \mu^{DA}$.

3. If each student is insufficiently sophisticated, in the original level-$k$ model I have shown that each $i$ must report a school strictly better than $\mu^{DA}(i)$ as first choice. In that model $i$ believes the others are $Lk_i - 1$. But in the informational level-$k$ model if there exists some $j$ such that $k_j < k_i - 1$, then $i$ knows $j$’s level. So in $i$’s belief in the informational level-$k$ model the market is weakly less competitive than
in i's belief in the original level-k model. Hence i must still report a school strictly better than $\mu^{DA}(i)$ as first choice. This implies that $\tilde{\mu}^{BM}_{ki}$ is not Pareto dominated by $\mu^{DA}$.

4. The proof of Lemma 2 implies that if $\tilde{\mu}^{BM}_{ki}$ is Pareto dominated by $\mu^{DA}$, there must exist some positive-level i who reports some $P'_i$ such that $\mu^{BM}_{ki}(i) P'_i \mu^{DA}(i)$ but $\mu^{DA}(i) P_i \mu^{BM}_{ki}(i)$. So if each positive-level i reports some $P'_i$ such that $\mu^{DA}(i) P'_i$, s for any s $\in S$, $\tilde{\mu}^{BM}_{ki}$ must not be Pareto dominated by $\mu^{DA}$.

**Proof of Proposition 8**

As proved before, if j is admitted by his reported first choice when being $L_{k_j}$, becoming a higher level does not change the outcome of BM. If j is rejected by his reported first choice when being $L_{k_j}$, let s be the school that finally admits j in BM. Then s must have empty seats after the first round of BM. In other words, j is unmatched in $\mu^{FDA^*}_{k_j}$ and s has empty seats in $\mu^{FDA^*}_{k_j}$. Now suppose j becomes $L_{k'_j}$ for any $k'_j > k_j$. To obtain the outcome of the first round of BM I study the outcome of Fast DA*. In Fast DA*, j will apply to some new schools after round $k_j$. Let the sequence of these new schools be $\{s_1, \ldots, s_v\}$. Since $k_j = \bar{k}_N$, no student in $N\{j\}$ will apply to new schools after round $k_j$ of Fast DA*. Hence to obtain the new outcome of Fast DA* I start with the old outcome $\mu^{FDA^*}_{k_j}$ and let j apply to the schools in the sequence one by one. When j applies to a school in the sequence, j is either rejected immediately, or is tentatively accepted and possibly induces a rejection-application chain.

Let $s_a$ be the first school in the sequence that accepts j tentatively. Since s has empty seats, s will accept j immediately if j applies to s. So $s_a$ must be weakly better than s. If $s_a$ finally accepts j, which means j is weakly better off, I finish the proof. If $s_a$ finally rejects j, then j must induce a rejection-application chain in which a student with a higher priority than j at $s_a$ applies to $s_a$ and replaces j. Since any student in $N\{j\}$ cannot apply to a new school if being rejected, the chain must not involve any student in $N\{j\}$. The chain must neither involve any school with empty seats. So after j being rejected by $s_a$, the only change in the outcome of Fast DA* is that some quasi-rational students exchange their seats. The set of unmatched students and the set of empty seats are same as before. Then I can repeat the above argument for $s_{a+1}$ and all following schools. If j is rejected by all schools in the sequence, those schools must be strictly
better than $s$, and $s_v$ is just the first choice reported by $j$ when being $Lk'_j$. So all schools weakly better than $s_v$ must be exhausted in the new outcome of Fast DA*, and the set of unmatched students and the set of empty seats in the new outcome of Fast DA* are same as before. Note that all unmatched students will report the same preferences as before since they cannot know the level change of $j$. Hence if $j$ uses a strategy satisfying the worse-rank condition, $j$ will apply to every school worse than $s_v$ in the same round of BM as before, and every other unmatched student will apply to the same school in the same round of BM as he did before. So $j$ must still be admitted by $s$. Hence I finish the proof.

B Pathak and Sönmez (2008) as Corollaries

In this appendix I prove that all results of Pathak and Sönmez (2008) are either implied by mine or can be proved easily through my method.

In any $P_I$, for any nonempty $N \subseteq I$ and nonempty $M = I \backslash N$ let all students in $N$ be naive and all students in $M$ be rational. I prove that the set of NE outcomes of BM in $P_I$ can be found by the following two-step procedure.

- Construct an artificial economy $(\{P^1_j\}_{j \in N}, \{P_\ell\}_{\ell \in M})$ where $P^1_j$ only lists the most preferred school of $j$. Let $\mathcal{M}_1$ be the set of stable matchings in this economy.

- For each $\mu \in \mathcal{M}_1$, finalize the assignments of all matched students, then run BM for unmatched students using their true preferences. Denote the matching found in this way by $f(\mu)$. Define $\mathcal{M}_2 \equiv \{f(\mu) : \mu \in \mathcal{M}_1\}$.

**Proposition 10.** $\mathcal{M}_2$ is the set of NE outcomes of BM when $N$ is naive and $M$ is rational.

**Proof.** By Ergin and Sönmez (2006), $\mathcal{M}_1$ is the set of NE outcomes of BM in the artificial economy if all students are rational. However, since each $j \in N$ is allowed to list only one school, $j$ cannot manipulate BM in the artificial economy. For each $\mu \in \mathcal{M}_1$, the students in $M$ obtain the same assignments in $\mu$ and $f(\mu)$. So it must be a NE of BM that each $j \in N$ reports $P_j$ and each $\ell \in M$ reports $\mu(\ell)$ as first choice in $P_I$. Hence $f(\mu)$ is a NE outcome of BM in $P_I$. 

40
Conversely, for each NE outcome $\mu$ in $P_I$, it must be a NE that each $j \in N$ reports $P_j$ and each $\ell \in M$ reports $\mu(\ell)$ as first choice. Then in the artificial economy it must still be a NE that each $j \in N$ reports $P^1_j$ and each $\ell \in M$ reports $\mu(\ell)$ as first choice. Note that the corresponding outcome of BM in the artificial economy must be $f^{-1}(\mu)$. Hence I finish the proof.

If I denote the student-optimal stable matching in the artificial economy by $\mu^*$, then $f(\mu^*)$ must be the student-optimal stable matching in $P_I$. So $f(\mu^*) = \mu^{DA}$. In the informational level-k model of BM, if $k_N = 0$, then all students in $N$ can only apply to their most preferred schools in Fast DA*. So the outcome of Fast DA* is just $\mu^*$. Note that this is the matching found by the first round of BM. So running the remaining procedures of BM is equivalent to the second step of the above two-stage procedure. So the outcome of BM is just $f(\mu^*) = \mu^{DA}$. This proves Proposition 9. So the comparative statics results of Pathak and Sönmez are implied by Proposition 5 and Proposition 6.

**Corollary 7.** (Propositions 3 and 4 of Pathak and Sönmez (2008)) Rational students are weakly better off in the student-optimal NE outcome of BM than in the outcome of DA. A naive student weakly benefits from becoming rational and all rational students weakly suffer.

Pathak and Sönmez also prove that each $j \in N$ obtains the same assignment in all NE outcomes of BM in $P_I$. I show that it is straightforwardly implied by the rural hospital theorem (Roth, 1986). Specifically, for each $\mu \in \mathcal{M}_1$, all students in $M$ must be matched in $\mu$. Each $j \in N$ is either matched to his most preferred school or unmatched in $\mu$. By the rural hospital theorem, if $j$ is matched in one stable matching, $j$ must be matched in all stable matchings. Hence if $j$ is matched in one $\mu$, $j$ must be matched to his most preferred school in all $\mu \in \mathcal{M}_1$, hence as well as in $\mathcal{M}_2$. On the other hand, by the rural hospital theorem each school must admit the same number of students in all $\mu \in \mathcal{M}_1$. Hence the number of unmatched students and the number of empty seats at each school are same in all $\mu \in \mathcal{M}_1$. Then at the second step of the above two-stage procedure each unmatched $j \in N$ must obtain the same assignment in all $\mu \in \mathcal{M}_2$.

**Corollary 8.** (Proposition 2 of Pathak and Sönmez (2008)) Each $j \in N$ is admitted by the same school in all NE outcomes of BM.
C Additional Simulation Results

In this section I present the additional simulation results. Figure 3 and Figure 4 show the results for the informational level-k model of BM in small and large markets respectively. Figure 5 and Figure 6 show the results for the original level-k model of BM in small and large unbalanced markets respectively.

D A Level-k Model of Constrained DA

Let $c < |S|$ be the length of the preference ordering a student can report. In this section I present the original level-k model of constrained DA by assuming that $L0$ students report their top $c$ choices and positive-level students use topping strategies.\(^{14}\)

Formally, let $P^c$ be the truncated version of any $P \in P$ that ranks the top $c$ choices. When each $i$ reports $P^c_i$, denote the outcome of DA by $\mu^c$. If $\mu^c(i) \neq \emptyset$, then $\mu^c(i)$ must

\(^{14}\)That is, they report their best obtained schools as first choices and report the true preference ordering of the remaining $c - 1$ top choices.
Figure 4: Informational level-k model of BM in large markets

Figure 5: Original Level-k model of BM in small unbalanced markets
Figure 6: Original Level-k model of BM in large unbalanced markets

be weakly better than \( \mu^{DA}(i) \). If all students are matched in \( \mu^c \), then \( \mu^c = \mu^{DA} \). Let \( s^k_i \) be the first choice reported by each \( i \) at \( L_k \), then I have the following result.

**Proposition 11.** For any \( P_i \), \( s^k_i \) \( R_i \) \( s^{k+1}_i \) \( R_i \) \( \mu^{DA}(i) \) for all \( i \in I \) and all \( k \geq 0 \). There exists some finite \( r_{i}^{DA} \geq 0 \) for each \( i \) such that \( s^k_i = \mu^{DA}(i) \) for all \( k \geq r_{i}^{DA} \).

**Proof.** At \( L_0 \) each \( i \) reports \( P_i^c \). So \( s^0_i \) is the most preferred school of \( i \). Denote the outcome of DA if all students are \( L_0 \) by \( \mu^0 \). It is obvious that \( \mu^0 = \mu^c \). If any school \( s \) admits \( q_s \) students in \( \mu^0 \), denote the priority rank of the lowest-priority admitted student by \( z^0_s \). Otherwise, define \( z^0_s \equiv |I| \). So \( z^0_s \) is the threshold of entering \( s \) in \( \mu^0 \). Let \( z_s^{DA} \) be the similar threshold in \( \mu^{DA} \), then it is obvious that \( z^0_s \geq z_s^{DA} \) for all \( s \).

At \( L_1 \), for each \( i \), if \( \mu^0(i) \neq \emptyset \), then \( \mu^0(i) \) is the best obtainable school for \( i \). Hence \( s^1_i = \mu^0(i) \). If \( \mu^0(i) = \emptyset \), then \( i \) will report a new best obtainable school \( s^1_i \) as first choice. Since \( z^0_s \geq z_s^{DA} \) for all \( s \), \( \mu^{DA}(i) \) must be obtainable for \( i \). So \( s^1_i \) must be weakly better than \( \mu^{DA}(i) \). Denote the outcome of DA if all students are \( L_1 \) by \( \mu^1 \). Denote the threshold of entering each \( s \) in \( \mu^1 \) by \( z^1_s \). Then \( z^1_s \leq z^0_s \) for all \( s \), which means all thresholds are weakly higher. By using topping strategies each \( i \) is either unmatched in \( \mu^1 \) or admitted by a school weakly better than \( \mu^{DA}(i) \). So \( z_s^{DA} \leq z^1_s \) for all \( s \).
At $Lk$ for any $k \geq 2$, suppose it is true that $s_{i}^{k-1} R_{i} s_{i}^{k'} R_{i} \mu^{DA}(i)$ and $z_{s}^{DA} \leq z_{s}^{k'} \leq z_{s}^{k'-1}$ for all $i$, all $s$ and all $k' < k$. Then for each $i$, any $s$ better than $s_{i}^{k-1}$ must be unobtainable for $i$. If $\mu^{k-1}(i) \neq \emptyset$, then $\mu^{k-1}(i)$ is the best obtainable school for $i$. If $\mu^{k-1}(i) = \emptyset$, $i$ will report a new first choice at $Lk$. But the school must be weakly better than $\mu^{DA}(i)$ since $z_{s}^{DA} \leq z_{s}^{k-1}$ for all $s$. Hence it is still true that $s_{i}^{k-1} R_{i} s_{i}^{k'} R_{i} \mu^{DA}(i)$ and $z_{s}^{DA} \leq z_{s}^{k} \leq z_{s}^{k-1}$ for all $i$ and all $s$. Then by induction, $s_{i}^{k} R_{i} s_{i}^{k+1} R_{i} \mu^{DA}(i)$ for all $i \in I$ and all $k \geq 0$.

When $k$ is high enough, all students at $Lk$ must be matched in $\mu^{k}$. Then $\mu^{k}$ must weakly Pareto dominate $\mu^{DA}$. Since $\mu^{k}(i)$ is the best obtainable school for each $i$ at $L(k + 1)$, $\mu^{k}$ must be stable. So $\mu^{k} = \mu^{DA}$. Hence there exists some finite $r_{i}^{DA} \geq 0$ for each $i$ such that $s_{i}^{k} = \mu^{DA}(i)$ for all $k \geq r_{i}^{DA}$.

Hence the above original level-k model of constrained DA looks like that of constrained BM. If all students have high enough levels, say quasi-rationality, the outcomes of both BM and DA are $\mu^{DA}$. But if some students have low levels, the comparison between BM and DA is ambiguous. Therefore I do simulations to compare them. I only consider large markets and choose $c = 5$. The simulation result is in Figure 7. The figure shows that constrained DA performs better than constrained BM, and higher-level students are more likely to have better assignments in both BM and DA. However, as said before a level-k model of constrained DA highly depends on the selection of students’ best strategies. By using topping strategies, positive-level students may report non-truthful rankings of the schools they report, and Calsamiglia, Haeringer and Klijn (2010) prove that these strategies are weakly dominated by truthful rankings of the schools they report. Hence to what an extent the above level-k model is credible is a question.

E Correct Proposition 1 of Abdulkadiroğlu, Che and Yasuda (2011)

Abdulkadiroğlu, Che and Yasuda (2011) prove their Proposition 1 in their on-line appendix which states that if students have common preferences in the strict priorities and complete information environment and some students are naive while the others are rational, then in the unique NE outcome of BM any naive student becoming rational will
Figure 7: Compare BM and DA in constrained school choice.

make all other naive students weakly worse off. Through the following example I show
that this statement is incorrect. Then I analyze when this statement holds.

Example 2. There are four schools \( \{s_1, s_2, s_3, s_4\} \) and four students \( \{i_1, i_2, i_3, i_4\} \). Each
school has only one seat. Students have the common preference ordering
\( s_1 \succ s_2 \succ s_3 \succ s_4 \). The priority rankings are shown below. If all students are naive, then they report true
preferences. Now suppose \( i_2 \) becomes rational, then he will report \( s_2 \) as first choice. The
outcomes of BM are shown below. It is easy to see that when \( i_2 \) becomes rational, \( i_2, i_4 \)
are better off, \( i_1 \) remains same, and \( i_3 \) is worse off.

![Priority rankings](image)

(a) Priority rankings  (b) All students are naive  (c) \( i_2 \) is rational, the others are naive

When some naive student \( j \) becomes rational, by Proposition 9 it is equivalent to
the situation that in the informational level-\( k \) model of BM \( k_N = 0 \), but some \( j \in N \)
becomes quasi-rational. Suppose there are \( m \) schools and the common preferences are 
\[ s_1 \succ s_2 \succ \cdots \succ s_m. \]
As proved before, if \( j \) is matched in Fast DA* when being \( L_0 \), becoming quasi-rational does not change the outcome of BM. If \( j \) is unmatched in Fast DA* when being \( L_0 \), let \( \{s_v, s_{v+1}, \ldots, s_m\} \) be the set of schools with empty seats in the outcome of Fast DA*. Let the number of empty seats of \( s_v \) be \( e_{s_v} \). Since students have common preferences, the number of empty seats of each \( s_a \) with \( a > v \) must be \( q_{s_a} \). Let \( s_u \) be the school that admits \( j \) in the outcome of BM. Hence \( u \in [v, m] \).

By becoming quasi-rational \( j \) must be matched in Fast DA*. So from the second round of BM on there will be one fewer empty seat and one fewer unmatched student than before. Since students have common preferences, the missing empty seat must belong to \( s_v \). Hence \( s_v \) has \( e_{s_v} - 1 \) empty seats now. So some student \( j_1 \) who was admitted by \( s_v \) in the second round of BM before will be rejected by \( s_v \) now. Then \( j_1 \) will apply to \( s_{v+1} \), and either be rejected or replace another student who was admitted by \( s_{v+1} \) before. Repeating this argument there must be exactly one student \( j_a \) who was admitted by some school in \( \{s_v, \ldots, s_{u-1}\} \) but now is rejected. Moreover, all the students who were admitted by the schools in \( \{s_v, \ldots, s_{u-1}\} \) before are weakly worse off and some is strictly worse off (e.g., \( j_1 \) and \( j_a \)). \( j_a \) will apply to \( s_u \). Since \( j \) is matched in the first round, compared to the before situation \( j_a \) essentially replaces \( j \) among those who applied to \( s_u \) before. Let \( j_b \) be the highest-priority student at \( s_u \) who was rejected by \( s_u \) before. If \( j_b \) has a higher priority than \( j_a \), then \( j_b \) will be admitted by \( s_u \) and hence be strictly better off (this is what happens in the above example). Then \( j_a \) will apply to \( s_{u+1} \) and essentially replace \( j_b \) among those who applied to \( s_{u+1} \) before. On the other hand, if \( j_b \) has a lower priority than \( j_a \) at \( s_u \), then \( j_a \) is admitted by \( s_u \). Then the assignments of all naive students who were admitted by the schools in \( \{s_{u+1}, \ldots, s_m\} \) do not change.